V. "On certain Geometrical Theorems. No. 1." By W. H. L. RUSSELL, F.R.S. Received November 12, 1881.

(1.) The following proof of the equation to a circle inscribed in a triangle, expressed in trilinear co-ordinates, is very short and simple.

Let α, β, γ be the sides of the triangle, A, B, C the opposite angles, and let

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0$$

be the equation to an inscribed conic. Then when this conic is a circle, the centre is given by the equations $\alpha=\beta=\gamma$, and the equation to the line joining the centre, to the point where γ touches the conic, that is to the point $l\alpha-m\beta=0$, $\gamma=0$, is

$$l\alpha-m\beta+(m-l)\gamma=0.$$

Now, when the conic is a circle, this line must be perpendicular to γ; hence from the condition that two straight lines may be perpendicular to each other (Salmon, "Conic Sections," 6th edition, Art. 61),

$$m-l=l\cos \mathrm{B}-m\cos \mathrm{A},$$

or

$$\frac{l}{\cos^2\frac{\mathrm{A}}{2}}=\frac{m}{\cos^2\frac{\mathrm{B}}{2}}=\frac{n}{\cos^2\frac{\mathrm{C}}{2}},$$

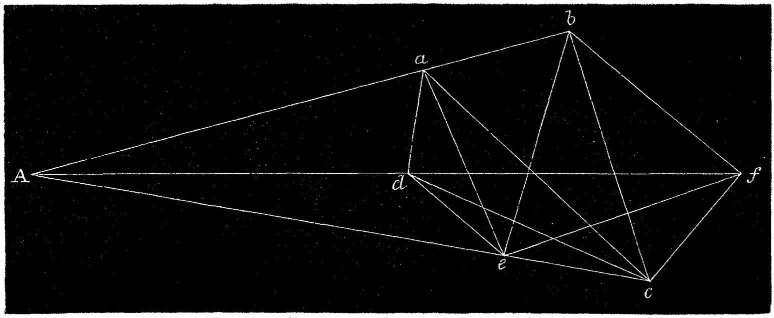
which gives for the required circle

$$\alpha^2\cos^4\frac{\mathrm{A}}{2}+\beta^2\cos^4\frac{\mathrm{B}}{2}+\gamma^2\cos^4\frac{\mathrm{C}}{2}$$
$$-2\beta\gamma\cos^2\frac{\mathrm{B}}{2}\cos^2\frac{\mathrm{C}}{2}-2\alpha\beta\cos^2\frac{\mathrm{A}}{2}\cos^2\frac{\mathrm{B}}{2}-2\alpha\gamma\cos^2\frac{\mathrm{A}}{2}\cos^2\frac{\mathrm{C}}{2}=0.$$

(2.) The following theorem is given by Dr. Salmon in his "Higher Plane Curves":—

If through any point of inflexion A in a curve of the third order there be drawn three right lines meeting the curve in *ab*, *df*, *ec*, then every curve of the third degree passing through the seven points A, *a*, *b*, *d*, *f*, *c*, *e* will have A for a point of inflexion. It follows from this that any curve of the third degree described through the nine points of inflexion of a cubic will have those points as points of inflexion.

Dr. Salmon has given a geometrical proof of this theorem, and this is the only demonstration I have ever seen. I have, therefore, obtained the following analytical proof, which possesses, I think, considerable beauty.



Let A be the origin, Adf the axis of (x), Aec the axis of y. Let $y=lx$ be the equation to Aab, $y=ax+b$ the equation to cd, $y=nx+b$ the equation to ca, $y=cx+e$ the equation to ef, $y=mx+e$ the equation to be, then we may find the equation to ad,

$$y(l-n+a)-lax-lb=0;$$

and similarly the equation to bf,

$$y(l-m+c)-lcx-le=0.$$

In this way it will easily be seen that the six points, a, b, c, d, e, f, are completely determined, and consequently the equation to a curve of the third degree passing through them (see Salmon, "Higher Plane Curves," Art. 162) is

$$ab\,.\,cd\,.\,ef+\theta\,.\,ac\,.\,be\,.\,df+\phi\,.\,ad\,.\,bf\,.\,ce+\psi\,.\,ae\,.\,bd\,.\,cf=0.$$

But since ab, df, ce pass through the origin $\psi=0$, and the equation becomes

$$ab\,.\,cd\,.\,ef+\theta\,.\,ac\,.\,be\,.\,df+\phi\,.\,ad\,.\,bf\,.\,ce=0,$$

and consequently writing down ab, cd, ef, ac, be, df, ad, bf, ce, as given above, we have as the equation of the required cubic—

$$(y-lx)(y-ax-b)(y-cx-c)+\theta y(y-nx-b)(y-mx-e)$$
$$+\phi x(y(l-n+a)-lax-lb)((l-m+c)y-lcx-le)=0;$$

differentiating this equation, and putting $x=y=0$ to determine the value of $\frac{dy}{dx}$ at the origin, we have

$$\frac{dy}{dx}(1+\theta)=l-\phi l^2.$$

Differentiating again, and putting $\frac{d^2y}{dx^2}=0$, $x=y=0$, since the origin

is to be a point of inflexion, we shall have—

$$\frac{dy^2}{dx^2}e-(l+a)e\frac{dy}{dx}+(l-\phi l^2)ae+\frac{dy^2}{dx^2}b-(l+c)b\frac{dy}{dx}+(l-\phi l^2)cb$$

$$+\theta e\frac{dy^2}{dx^2}-ne\theta\frac{dy}{dx}+\theta b\frac{dy^2}{dx^2}-mb\theta\frac{dy}{dx}+\phi el(l-n+a)\frac{dy}{dx}+\phi bl(l-m+c)\frac{dy}{dx}$$

or substituting for $l-\phi l^2$, and dividing by $\frac{dy}{dx}$, we have—

$$\frac{dy}{dx}e-(l+a)e+(1+\theta)ae+\frac{dy}{dx}.b-(l+c)b+(1+\theta)cb$$
$$+\theta c\frac{dy}{dx}-ne\theta+\theta b\frac{dy}{dx}-mb\theta+\phi(l-n+a)el+\phi(l-m+c)bl=0.$$

Again substituting $l-\phi l^2$ for $(1+\theta)\frac{dy}{dx}$, and reducing, we obtain the equation

$$(\theta+l\phi)(ae+cb-ne-mb)=0.$$

Hence, if $ae+cb-ne-mb$ vanish, the origin will be a point of inflexion, whatever values we give to θ and ϕ; hence the theorem is true.

(3.) From any point six tangents can be drawn to a curve of the third order; two of these are at right angles to one another, determine the locus of the point.

Substitute for y in the general equation of the cubic $m(x-\xi)+\eta$, arrange the terms of the resulting equation according to powers of (x) and form the discriminant, equate the discriminant to zero, and we shall have an equation of the form—

$$m^6-am^5+bm^4-cm^3+dm^2-em+f=0.$$

Let $$m_1+m_2+m_3+m_4+m_5+m_6=a,$$

$$m_1m_2+(m_1+m_2)(m_3+m_4+m_5+m_6)+m_3m_4+m_3m_5+m_3m_6+m_4m_5$$
$$+m_4m_6+m_5m_6=b,$$

$$m_1m_2(m_3+m_4+m_5+m_6)$$
$$+(m_1+m_2)(m_3m_4+m_3m_5+m_3m_6+m_4m_5+m_4m_6+m_5m_6)$$
$$+m_3m_4m_5+m_3m_4m_6+m_3m_5m_6+m_4m_5m_6=c,$$

$$m_1m_2(m_3m_4+m_3m_5+m_3m_6+m_4m_5+m_4m_6+m_5m_6)$$
$$+(m_1+m_2)(m_3m_4m_5+m_3m_4m_6+m_3m_5m_6+m_4m_5m_6)+m_3m_4m_5m_6=d.$$

$$m_1m_2(m_3m_4m_5+m_3m_4m_6+m_3m_5m_6+m_4m_5m_6)$$
$$+(m_1+m_2)(m_3m_4m_5m_6)=e,$$

$$m_1m_2m_3m_4m_5m_6=f.$$

Since two of the tangents are at right angles to each other, we shall have $m_1m_2+1=0$, and let $m_1+m_2=\mu$, $\Sigma m_3=p$, $\Sigma m_3m_4=q$, $\Sigma m_3m_4m_5=r$, $m_3m_4m_5m_6=s$. Then substituting, we have the following equations:—

$$\mu+p=a \quad \ldots \quad (1), \qquad -q+\mu r+s=d \quad \ldots \quad (4),$$
$$-1+\mu p+q=b \quad \ldots \quad (2), \qquad -r+\mu s=e \quad \ldots \quad (5),$$
$$-p+\mu q+r=c \quad \ldots \quad (3), \qquad -s=f \quad \ldots \quad (6).$$

From these equations we obtain at once—

$$\mu=a-p. \qquad r=f(p-a)-e, \qquad q=1+b-\mu p=1+b-ap+p^2.$$

Hence we have, substituting in (4)—

$$(f+1)p^2-(a+2af+e)p+(1+b)+fa^2+ae+f+d=0 \quad . \quad . \quad (7).$$

Also substituting in (3)—

$$p^3-2ap^2+(a^2+b-f+2)p-a(b-f+1)+e+c=0 \quad . \quad . \quad (8).$$

From (7) and (8) we easily obtain two equations of the form—

$$\mathrm{A}p^2+\mathrm{B}p+\mathrm{C}=0,$$
$$\mathrm{A}'p^2+\mathrm{B}'p+\mathrm{C}'=0,$$

then the eliminant is at once seen to be—

$$(\mathrm{A}'\mathrm{C}-\mathrm{C}'\mathrm{A})^2+(\mathrm{BA}'-\mathrm{AB}')(\mathrm{BC}'-\mathrm{CB}')=0,$$

the equation to the required locus.

I have not thought it necessary to write down the values of a, b, c, &c., as they are obtained by rules perfectly well known.

Note by W. Spottiswoode, P.R.S.

The second theorem in the foregoing paper follows also as an immediate consequence of a formula given by Cayley in his "Seventh Memoir on Quantics" ("Phil. Trans.," 1861, p. 286). If U represent the cubic and HU its Hessian, then, as is well known, HU passes through the points of inflexion of U. Also, the function $\alpha\mathrm{U}+6\beta\mathrm{HU}$ will represent an arbitrary curve of the third degree passing through the same points; and, on the same principle as before, its Hessian will pass through its points of inflexion. Now the formula in question is—

$$\mathrm{H}(\alpha\mathrm{U}+6\beta\mathrm{HU})= \quad \delta_\beta(1,0,-24\mathrm{S},\ldots)(\alpha,\beta)^4\,.\,\mathrm{U}$$
$$\qquad\qquad -6\delta_\alpha(1,0,-24\mathrm{S},\ldots)(\alpha,\beta)^4\,.\,\mathrm{HU}.$$

But this equation is satisfied by $U=0$, $HU=0$; consequently the equations

$$\alpha U+6\beta HU=0, \qquad H(\alpha U+6\beta HU)=0,$$

are both satisfied by the relations

$$U=0, \qquad HU=0.$$

Hence the theorem given in the text.

VI. "On a Class of Invariants." By JOHN C. MALET, M.A., Professor of Mathematics, Queen's College, Cork. Communicated by Professor CAYLEY, LL.D., F.R.S. Received December 14, 1881.

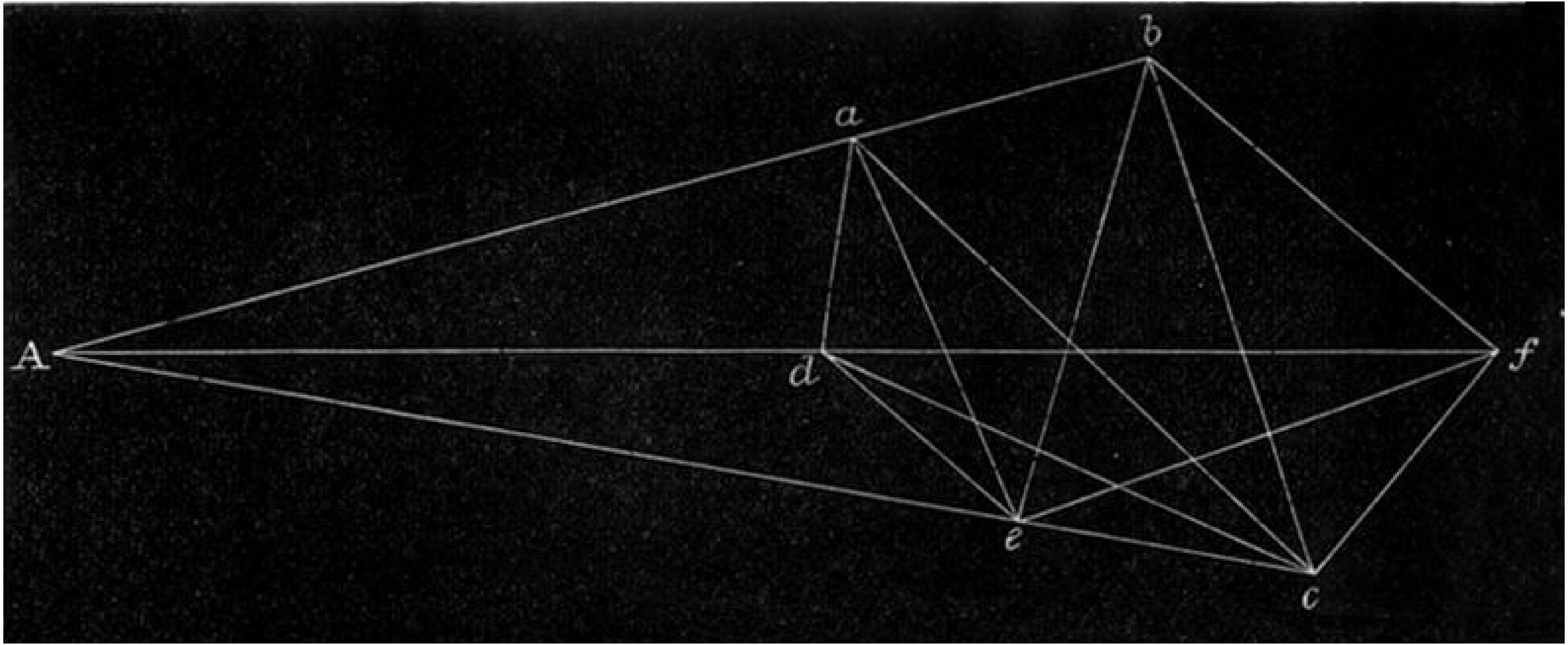
(Abstract.)

This paper is concerned with two kinds of functions of the coefficients of Linear Differential Equations, which have certain invariant properties.

In the first part of the paper it is shown that every Linear Differential Equation possesses a certain number of functions of the coefficients which are unaltered by changing the dependent variable y to yu where u is any given function of x, the independent variable. These functions bear remarkable analogies to functions of the differences of the roots of ordinary algebraic equations, and many problems, provided they involve only the ratios of the solutions of the differential equation, may be solved in terms of them; for example, the condition that two solutions y_1 and y_2 of a linear differential equation of the third order should be connected by the relation $y_1=y_2x$ is expressed in terms of two such functions of the coefficients of the equation. This problem is analogous to that of finding the discriminant of an algebraic binary cubic.

The second part of the paper is concerning functions of the coefficients of Linear Differential Equations which are unaltered by change of the independent variable, and the theory of these functions is applied to the solutions of problems involving only relations among the solutions of the equation without the independent variable.

In this part of the paper it is shown how to form the condition that the three solutions y_1, y_2, y_3 of a linear differential equation of the third order should be connected by the relation $y_1y_2=y_3^2$, which relation, involving only ratios of the solutions, and not containing the independent variable, can be expressed in terms of either class of the functions of the coefficients considered in the paper; these two methods of writing the condition are accordingly given.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