

April 30, 1885.

THE PRESIDENT in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read :—

- I. “Abstract of some Results in Elliptic Functions. (Part II.)”
By JOHN GRIFFITHS, M.A. Communicated by Professor
G. G. STOKES, Sec. R.S. Received April 9, 1885.

1. *On the $z(u)$ Function Complementary to Jacobi's $Z(u)$.*

The double periodicity of the elliptic functions gives rise to an interesting function of the form $z(u) = a - \frac{J'}{K'}u$, where

$$u = \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad a = \int_0^\theta \sqrt{1-k^2\sin^2\theta} d\theta = E(u), \quad J' = K' - E'.$$

By changing a, u , respectively, into (1) $a + 2iJ', u + 2iK'$, (2) $a + 2E, u + 2K$, (3) $a + E - k^2 \frac{snucnu}{dn u}, u + K$, it is easily seen that $z(u)$ satisfies the following relations, viz. :—

$$z(u + 2iK') = z(u),$$

$$z(u + 2K) - z(u) = \frac{\pi}{K'} \quad (\text{since } KE' + K'E - KK' = \frac{\pi}{2}),$$

$$z(u + K) - z(u) = \frac{\pi}{2K'} - k^2 \frac{snucnu}{dn u}.$$

2. *Deduction of a $\Phi(u)$ Function from $z(u)$.*

Writing $\Phi(u) = \sqrt{\frac{2k'K'}{\pi}} e^{\int_0^{z(u)} du}$ we can take the foregoing $z(u)$ relations as equivalent to—

$$\Phi(u + 2iK') = -\Phi(u),$$

$$\Phi(u + 2K) = \frac{1}{i} e^{\frac{\pi u}{K}} \Phi(u),$$

$$dn u = \sqrt{k'} r^{\frac{1}{2}} e^{-\frac{\pi u}{2K'}} \frac{\Phi(u+K)}{\Phi(u)},$$

where $r = e^{-\frac{\pi K}{K'}}$

$\Phi(u)$ is, in fact, connected with Jacobi's $\Theta(u)$ by the equation
 $\Phi(u) \div \Phi(0) = e^{-\frac{\pi u^2}{4KK'}} \Theta(u) \div \Theta(0).$

3. *Expansion of $\Phi(u)$ in a Hyper-harmonic Series containing odd Multiples of $\frac{\pi u}{2K'}$.*

From the above materials it is found that—

$$\Phi(u) = 2 \left\{ \sqrt[4]{r} \cosh \frac{\pi u}{2K'} + \sqrt[4]{r^9} \cosh \frac{3\pi u}{2K'} + \sqrt[4]{r^{25}} \cosh \frac{5\pi u}{2K'} + \dots \text{ad infn.} \right\},$$

where $r = e^{-\frac{\pi K}{K'}}$ and $\cosh x = \frac{1}{2}(e^x + e^{-x}).$

4. *Some Consequences of the above Theorems.*

Among the numerous results which flow from the above I notice the following, viz. :—

$$(a.) \quad a - \frac{J'}{K'} u = \frac{\pi}{2K'} \frac{\sqrt[4]{r} \sinh \frac{\pi u}{2K'} + 3\sqrt[4]{r^9} \sinh \frac{3\pi u}{2K'} + \dots}{\sqrt[4]{r} \cosh \frac{\pi u}{2K'} + \sqrt[4]{r^9} \cosh \frac{3\pi u}{2K'} + \dots}.$$

If this be combined with Jacobi's

$$Z(u) = a - \frac{E}{K} u = \frac{2\pi}{K} \left\{ \frac{q}{1-q^3} \sin \frac{\pi u}{K} + \dots \right\}$$

we have the curious relation—

$$\frac{u}{*KK'} = \frac{1}{K'} \frac{\sqrt[4]{r} \sinh \frac{\pi u}{2K'} + 3\sqrt[4]{r^9} \sinh \frac{3\pi u}{2K'} + 5\sqrt[4]{r^{25}} \sinh \frac{5\pi u}{2K'} + \dots}{\sqrt[4]{r} \cosh \frac{\pi u}{2K'} + \sqrt[4]{r^9} \cosh \frac{3\pi u}{2K'} + \sqrt[4]{r^{25}} \cosh \frac{5\pi u}{2K'} + \dots}$$

* Other relations follow from the z function $z(u) = a - \frac{E + iJ'}{K + iK'} u$, which deserves to be studied. As regards the *transformation* of the function Φ , the results are very similar to those obtained in the case of Jacobi's Θ .—April 29, 1885.

$$-\frac{4}{K} \left\{ \frac{q}{1-q^2} \sin \frac{\pi u}{K} + \frac{q^2}{1-q^4} \sin \frac{2\pi u}{K} + \frac{q^3}{1-q^6} \sin \frac{3\pi u}{K} + \dots \right\}.$$

(β.) Putting $u=2nK$, we deduce a simple identity, viz.:—

If n be an integer, then—

$$(1+2n)r^n + (3+2n)r^{2+3n} + (5+2n)r^{6+5n} + (7+2n)r^{12+7n} + \dots \text{ad infin.} \\ = (1-2n)r^{-n} + (3-2n)r^{2-3n} + (5-2n)r^{6-5n} + (7-2n)r^{12-7n} + \dots \text{ad infin.}$$

(γ.) From the formula $\text{dn}u = \sqrt{k'} r^{\frac{1}{2}} e^{-\frac{\pi u}{2K'}} \frac{\Phi(u+K)}{\Phi(u)}$,

we have

$$\begin{aligned} \sqrt{k'} &= \frac{\Phi(0)}{r^{\frac{1}{2}} \Phi(K)} \\ &= \frac{\{2\sqrt[4]{r} + \sqrt[4]{r^9} + \sqrt[4]{r^{25}} + \sqrt[4]{r^{49}} + \dots\}}{r^{\frac{1}{2}}(r^{\frac{1}{2}} + r^{-\frac{1}{2}}) + r^{\frac{3}{2}}(r^{\frac{3}{2}} + r^{-\frac{3}{2}}) + r^{\frac{5}{2}}(r^{\frac{5}{2}} + r^{-\frac{5}{2}}) + \dots} \\ &= 2 \frac{\sqrt[4]{r} + \sqrt[4]{r^9} + \sqrt[4]{r^{25}} + \sqrt[4]{r^{49}} + \dots}{1 + 2r + 2r^4 + 2r^9 + 2r^{16} + \dots} \end{aligned}$$

This result is, in fact, Jacobi's

$$\sqrt{k} = 2 \frac{\sqrt[4]{q} + \sqrt[4]{q^9} + \sqrt[4]{q^{25}} + \dots}{1 + 2q + 2q^4 + \dots}, \text{ as we can see by changing } k^r$$

into k , and consequently r into q .

(δ.) From $\Phi(u) = \sqrt{\frac{2k'K'}{\pi}} e^{\int_0^u z(u) du}$ we deduce

$$\begin{aligned} \sqrt{\frac{2k'K'}{\pi}} &= \Phi(0) \\ &= 2\{\sqrt[4]{r} + \sqrt[4]{r^9} + \sqrt[4]{r^{25}} + \sqrt[4]{r^{49}} + \dots\} \\ \text{i.e., } \sqrt{\frac{2K'}{\pi}} &= 1 + 2r + 2r^4 + 2r^9 + 2r^{16} + \dots \end{aligned}$$

5. Extension of the above Method to a $\xi_1(u)$ Function connected with Elliptic Integrals of the Third Kind.

In a former note by the present writer mention was made of a $\zeta(u)$ function of the form $\zeta(u) = \frac{\pi}{2(1-\mu)\Pi K'} \left(p - \frac{\Pi}{K}\right)$,

where

$$p = \int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \quad \Pi = \int_0^\pi \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},$$

$$\mu = \frac{P'}{\Pi} \div \frac{K'}{K}, \quad P' = K' - \frac{n}{1+n} \Pi',$$

$$\Pi' = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+n'\sin^2\theta)\sqrt{1-k'^2\sin^2\theta}}, \quad n'(1+n) = -k'^2.$$

This is not exactly the form considered by Jacobi, but if we write $\frac{\Pi}{K} - 1 = \frac{\text{tn}u_0}{\text{dn}u_0} Z(u_0)$ and $n = -k^2sn^2u_0$ his result is equivalent to

$$\zeta(u) = \frac{1}{2u_0} \log \frac{\Theta(u+u_0)}{\Theta(u-u_0)}$$

Connected with $\zeta(u)$ is a *second* function of the form

$$\zeta_1(u) = \frac{\pi}{2(1-\mu)\Pi K'} \left(p - \frac{P'}{K'} u \right).$$

This satisfies the relations

$$\left. \begin{aligned} \zeta_1(u+2iK') &= \zeta_1(u) \\ \zeta_1(u+2K) - \zeta_1(u) &= \frac{\pi}{K'} \end{aligned} \right\},$$

and I find that it can be expressed in terms of $\Phi(u)$ by means of the equation $\zeta_1(u) = \frac{1}{2u_0} \log \frac{\Phi(u+u_0)}{\Phi(u-u_0)}$, where u_0 is the same constant as above.

It thus appears that $\Theta(u)$ and $\Phi(u)$ are connected with and supplement each other in a very remarkable manner.

For example, if we write $\zeta(u)$ and $\zeta_1(u)$ in the more convenient forms $\zeta(u, u_0)$, $\zeta_1(u, u_0)$, it follows that besides Jacobi's result, $u_0\zeta(u, u_0) = u\zeta(u_0, u)$, we have likewise the equivalent form $u_0\zeta_1(u, u_0) = u\zeta_1(u_0, u)$.

II. "Further Observations on Enterochlorophyll and Allied Pigments." By C. A. MACMUNN, M.A., M.D. Communicated by Professor M. FOSTER, Sec. R.S. Received April 21, 1885.

(Abstract.)

In a paper read before the Royal Society in 1883, I described the results of an examination of the so-called "bile" of invertebrates, and showed that the alcohol extracts of their liver or other appendage of the intestine answering to that organ, showed a spectrum so like that of vegetable chlorophyll, as to have led me to assume that no essential difference exists between the spectrum of enterochlorophyll and plant chlorophyll.