

February 3, 1887.

Professor STOKES, D.C.L., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read :—

- I. “On the Waves produced by a Single Impulse in Water of any Depth, or in a Dispersive Medium.” By Sir W. THOMSON, Knt., LL.D., F.R.S. Received January 26, 1887.

For brevity and simplicity consider only the case of *two-dimensional motion*.

All that it is necessary to know of the medium is the relation between the wave-velocity and the wave-length of an endless procession of periodic waves. The result of our work will show us that the velocity of progress of a zero, or maximum, or minimum, in any part of a varying group of waves, is equal to the velocity of progress of periodic waves of wave-length equal to a certain length, which may be defined as the wave-length in the neighbourhood of the particular point looked to in the group (a length which will generally be intermediate between the distances from the point considered to its next-neighbour corresponding points on its two sides).

Let $f(m)$ denote the velocity of propagation corresponding to wave-length $2\pi/\lambda$. The Fourier-Cauchy-Poisson synthesis gives

$$u = \int_0^{\infty} dm \cos m[x - tf(m)] \quad . \quad . \quad . \quad (1)$$

for the effect at place and time (x, t) of an infinitely intense disturbance at place and time $(0, 0)$. The principle of interference as set forth by Prof. Stokes and Lord Rayleigh in their theory of group-velocity and wave-velocity suggests the following treatment for this integral :—

When $x - tf(m)$ is very large, the parts of the integral (1) which lie on the two sides of a small range, $\mu - \alpha$ to $\mu + \alpha$, vanish by annulling interference; μ being a value, or the value, of m , which makes

$$\frac{d}{dm} \{m[x - tf(m)]\} = 0; \quad . \quad . \quad . \quad (2)$$

so that we have $x = t\{f(\mu) + \mu f'(\mu)\} = yt, \dots (3)$

where $y = f(\mu) + \mu f'(\mu);^* \dots (4)$

and we have by Taylor's theorem for $m - \mu$ very small :

$$m[x - tf(m)] = \mu[x - tf(\mu)] - \frac{1}{2}t[\mu f''(\mu) + 2f'(\mu)](m - \mu)^2, \dots (5)$$

or, modifying by (3)

$$m[x - tf(m)] = t\{\mu^2 f'(\mu) + \frac{1}{2}[-\mu f''(\mu) - 2f'(\mu)](m - \mu)^2\}. \dots (6)$$

Put now $m - \mu = \frac{\sigma \sqrt{2}}{t^{\frac{1}{2}}[-\mu f''(\mu) - 2f'(\mu)]^{\frac{1}{2}}}, \dots (7)$

and using the result in (1), we find

$$u = \frac{\sqrt{2} \int_{-\infty}^{\infty} d\sigma \cos [t\mu^2 f'(\mu) + \sigma^2]}{t^{\frac{1}{2}}[-\mu f''(\mu) - 2f'(\mu)]^{\frac{1}{2}}}; \dots (8)$$

the limits of the integral being here $-\infty$ to ∞ , because the denominator of (7) is so infinitely great that, though $\pm \alpha$, the arbitrary limits of $m - \mu$, are infinitely small, α multiplied by it is infinitely great.

Now we have $\int_{-\infty}^{\infty} d\sigma \cos \sigma^2 = \int_{-\infty}^{\infty} d\sigma \sin \sigma^2 = \sqrt{(\frac{1}{2}\pi)} \dots (9)$

Hence (8) becomes

$$u = \frac{\cos [t\mu^2 f'(\mu)] - \sin [t\mu^2 f'(\mu)]}{t^{\frac{1}{2}}[-\mu f''(\mu) - 2f'(\mu)]^{\frac{1}{2}}} = \frac{\sqrt{2} \cos [t\mu^2 f'(\mu) + \frac{1}{4}\pi]}{t^{\frac{1}{2}}[-\mu f''(\mu) - 2f'(\mu)]^{\frac{1}{2}}}. (10)$$

To prove the law of wave-length and wave-velocity for any point of the group, remark that, by (3)

$$t\mu^2 f'(\mu) = \mu[x - tf(\mu)],$$

and therefore the numerator of (10) is equal to $\sqrt{2} \cos \theta$, where

$$\theta = \mu[x - tf(\mu)] + \frac{1}{4}\pi, \dots (10')$$

and by (2) and (3) $\frac{d}{d\mu}\{\mu[x - tf(\mu)]\} = 0;$

by which we see that

$$d\theta/dx = \mu, \quad \text{and} \quad d\theta/dt = -\mu f(\mu), \dots (10'')$$

which proves the proposition.

* This is the group-velocity according to Lord Rayleigh's generalisation of Prof. Stokes's original result.

Example (1).—As a first example take deep-sea waves; we have

$$f(m) = \sqrt{\frac{g}{m}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which reduces (4), (3), and (10) to

$$y = \frac{1}{2} \sqrt{\frac{g}{\mu}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$x = \frac{1}{2} \sqrt{\frac{g}{\mu}} \cdot t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$u = \frac{1}{g\sqrt{2}} \frac{t}{x^{\frac{3}{2}}} \left(\cos \frac{gt^2}{4x} + \sin \frac{gt^2}{4x} \right) = \frac{t}{gx} \cos \left(\frac{gt^2}{4x} - \frac{\pi}{4} \right), \quad . \quad (14)$$

which is Cauchy and Poisson's result for places where x is very great in comparison with the wave-length $2\pi/\mu$, that is to say, for place and time such that $gt^2/4x$ is very large.

Example (2).—Waves in water of depth D :

$$f(m) = \sqrt{\left\{ \frac{g}{m} \frac{1 - \epsilon^{-2mD}}{1 + \epsilon^{-2mD}} \right\}}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Example (3).—Light in a dispersive medium.

Example (4).—Capillary gravitational waves:

$$f(m) = \sqrt{\left(\frac{g}{m} + Tm \right)}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Example (5).—Capillary waves:

$$f(m) = \sqrt{(Tm)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Example (6).—Waves of flexure running along a uniform elastic rod:

$$f(m) = m \sqrt{\frac{B}{w}}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where B denotes the flexural rigidity, and w the mass per unit of length.

These last three examples have been taken by Lord Rayleigh as applications of his generalisation of the theory of group-velocity; and he has pointed out in his "Standing Waves in Running Water" (London Mathematical Society, December 13, 1883) the important peculiarity of Example (4) in respect to the critical wave-length which gives minimum wave-velocity, and therefore group-velocity equal to wave-velocity. The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interest-

ing, but time does not permit its being included in the present communication.

For Examples (5) and (6) the denominator of (10) is imaginary; and the proper modification, from (7) forwards, gives for these and such cases, instead of (14), the following:—

$$u = \frac{\cos [t\mu^2 f'(\mu)] + \sin [t\mu^2 f'(\mu)]}{t^{\frac{1}{2}} [\mu f''(\mu) + 2f'(\mu)]^{\frac{1}{2}}}. \quad \dots \quad (19)$$

The result is easily written down for each of the two last cases [Examples (5) and (6)].

II. "On the Formation of Coreless Vortices by the Motion of a Solid through an inviscid incompressible Fluid." By Sir W. THOMSON, Knt., LL.D., F.R.S. Received February 1, 1887.

Take the simplest case: let the moving solid be a globe, and let the fluid be of infinite extent in all directions. Let its pressure be of any given value, P , at infinite distances from the globe, and let the globe be kept moving with a given constant velocity, V .

If the fluid keeps everywhere in contact with the globe, its velocity relatively to the globe at the equator (which is the place of greatest relative velocity) is $\frac{3}{2}V$. Hence, unless $P > \frac{5}{8}V^2$,* the fluid will not remain in contact with the globe.

Suppose, in the first place, P to have been $> \frac{5}{8}V^2$, and to be suddenly reduced to some constant value $< \frac{5}{8}V^2$. The fluid will be thrown off the globe at a belt of a certain breadth, and a violently disturbed motion will ensue. To describe it, it will be convenient to speak of velocities and motions *relative to the globe*. The fluid must, as indicated by the arrow-heads in fig. 1, flow partly backwards and partly forwards, at the place, I , where it impinges on the globe, after having shot off at a tangent at A . The back-flow along the belt that had been bared must bring to E some fluid; and the free surface of this fluid must collide with the surface of the fluid leaving the globe at A . It might be supposed that the result of this collision would be a "vortex sheet," which in virtue of its instability, would get drawn out and mixed up indefinitely, and be carried away by the fluid farther and farther from the globe. A definite amount of kinetic energy would be *practically annulled* in a manner which I hope to explain in an early communication to the Royal Society of Edinburgh.

But it is impossible, either in our ideal inviscid incompressible

* The density of the fluid is taken as unity.