V. "On certain Definite Integrals. No. 16." By W. H. L. Russell, F.R.S. Received May 31, 1888.

In these papers I have considered incidentally the advantages gained by differentiating and integrating with regard to the quantities which are independent of the leading variable. In the present communication I enter upon this subject more systematically, as it evidently admits of wide extension.

$$\int_0^\omega \frac{\epsilon^{-a^2x^2}dx}{m^2+x^2} = \frac{\pi\epsilon^{m^2a^2}}{m}\int_{am}^\omega \epsilon^{-x^2}\,dx.$$

$$\int_0^n \{\epsilon^{3i\theta}\phi\epsilon^{2i\theta} + \epsilon^{-3i\theta}\phi\epsilon^{-2i\theta}\}\frac{d\theta}{\sin\theta} = 2i\pi\phi 0.$$

(See No. 88 of this series.)

$$\int_0^{\frac{n}{2}} d\theta\cos\theta\left\{\frac{\epsilon^{i\tan\theta+\theta}}{1-\mu\epsilon^{i\sin\theta}}\cdot\phi\frac{\epsilon^{i\tan\theta}}{1-\mu\epsilon^{i\tan\theta}} + \right.$$

$$\left.\frac{\epsilon^{-i(\tan\theta+\theta)}}{1-\mu\epsilon^{-i}\tan\theta}\phi\frac{\epsilon^{-i\tan\theta}}{1-\mu\epsilon^{-i\tan\theta}}\right\} = \frac{\pi}{2}\cdot\frac{1}{\epsilon-\mu}\phi\frac{1}{\epsilon-\mu}.$$

This is obtained from integral 21 of this series, where, however, in the denominators of the integral cos tan θ is misprinted cos θ.

$$\int_0^{\frac{n}{2}} dx\left\{\frac{\cos x\epsilon^{i(\tan x+x)}}{1-\alpha\cos x\epsilon^{ix}}\phi\left(\frac{\cos x\epsilon^{ix}}{1-\alpha\cos x\epsilon^{ix}}\right)\right.$$

$$\left.+\frac{\cos x\epsilon^{-i(\tan x+x)}}{1+\alpha\cos x\epsilon^{-ix}}\phi\left(\frac{\cos x\epsilon^{-ix}}{1-\alpha\cos x\epsilon^{-ix}}\right)\right\} = \frac{\pi}{\epsilon^c}\frac{1}{2-\alpha}\phi\frac{1}{2-\alpha}.$$

See integral 22.

$$\int_0^{\frac{n}{2}}\frac{d\theta}{\sin\theta}\{\epsilon^{i\theta}\phi(\cos\theta\epsilon^{i\theta}) + \epsilon^{-i\theta}\phi(\cos\theta\epsilon^{-i\theta})\} = \pi i\phi(1).$$

See Abel, 'Œuvres Complètes,' vol. 2, page 88.

$$\int_0^{\frac{n}{2}} d\theta\,\{\cos^2\theta\;\epsilon^{2x\cos^2\theta}\;\epsilon^{ix\sin 2\theta}\;\phi\;(2\cos\theta\;\epsilon^{2\theta})$$

$$+\cos^2\theta\;\epsilon^{2x\cos^2\theta}\epsilon^{-ix\sin 2\theta}\phi\;(2\cos\theta\;\epsilon^{-i\theta})\}$$

$$= \pi(2\phi(1) - \phi'(1)\epsilon^x + \phi(1)\,.\,x\epsilon^x).$$

See integral 116 of this series.

Again, since

$$\int_0^{\pi} d\,.\,\theta \frac{f\epsilon^{\theta i} + f\epsilon^{-\theta i}}{1 - 2\alpha\cos\theta + \alpha^2} = \frac{2\pi}{1-\alpha^2} f(\alpha),$$

we may write if $f(\alpha)$ be a rational fraction

$$\int_0^{\pi} d\theta \frac{f\epsilon^{\theta i} + f\epsilon^{-\theta i}}{\epsilon^{-\theta i} - \epsilon^{\theta i}} \left(\frac{1}{\epsilon^{\theta i} - \alpha} - \frac{1}{\epsilon^{-\theta i} - \alpha}\right) = \Sigma \frac{\mathrm{M}}{\mu - \alpha},$$

and, therefore,

$$\int_0^{n} d\theta \frac{f\epsilon^{\theta i} - f\epsilon^{-\theta i}}{\epsilon^{-\theta} - \epsilon^{\theta i}} \left\{\frac{1}{\epsilon^{\theta i} - \alpha}\,\phi\,\frac{1}{\epsilon^{\theta i} - \alpha}\right.$$

$$\left. - \frac{1}{\epsilon^{-\theta i} - \alpha}\,\phi\,\frac{1}{\epsilon^{\theta i} - \alpha}\right\} = \Sigma \frac{\mathrm{M}}{\mu - \alpha}\,\phi\,\frac{1}{\mu - \alpha}.$$

We know that

$$\int_0^{\omega} \frac{dx}{1+x^2} \frac{1}{1 - 2a\cos x + a^2} = \frac{\pi}{2} \frac{1}{1-a^2} \frac{\epsilon + a}{\epsilon - a},$$

that is—

$$\int_0^{\omega} \frac{dx}{1+x^2} \cdot \frac{1}{\epsilon^{-ix} - \epsilon^{ix}} \left\{\frac{1}{\epsilon^{ix} - a} - \frac{1}{\epsilon^{-ix} - a}\right\}$$

$$= \frac{\pi}{4} \cdot \frac{\epsilon + 1}{\epsilon - 1} \cdot \frac{1}{1-a} + \frac{\pi}{4} \frac{\epsilon - 1}{\epsilon + 1} \cdot \frac{1}{1+a} - \frac{\pi\epsilon}{\epsilon^2 - 1} \cdot \frac{1}{\epsilon - a}.$$

Hence we have

$$\int_0^{\omega} \frac{dx}{1+x^2} \cdot \frac{1}{\epsilon^{-ix} - \epsilon^{ix}} \left\{\frac{1}{\epsilon^{ix} - a}\,\phi\,\frac{1}{\epsilon^{ix} - a}\right.$$

$$\left. - \frac{1}{\epsilon^{-x} - a}\,\phi\,\frac{1}{\epsilon^{ix} - a}\right\} = \frac{\pi}{4} \cdot \frac{\epsilon + 1}{\epsilon - 1} \cdot \frac{1}{1-a}\,\phi\,\frac{1}{1-a}$$

$$+ \frac{\pi}{4} \frac{\epsilon - 1}{\epsilon + 1} \cdot \frac{1}{1+a}\,\phi\,\frac{-1}{1+a} - \frac{\pi\epsilon}{\epsilon^2 - 1} \cdot \frac{1}{\epsilon - a} \cdot \phi\,\frac{1}{\epsilon - a}.$$

Let now $\qquad \phi(x) = \left\{x^2 \dfrac{d^3}{dx^3} + 3x \dfrac{d^2}{dx^2} + \dfrac{d}{dx}\right\} \chi x$

be a relation connecting the two functions $\phi(x)$ and $\chi(x)$.

Then $x\phi(x) = \left(x\dfrac{d}{dx}\right)^3 \chi(x)$, and we may put $x\phi(x) = \mathrm{A}_0 x + \mathrm{A}_1 x^2 + \ldots + \mathrm{A}_n x^{n+1} + \ldots$, then making use of the symbol $\left(x\dfrac{x}{dx}\right)^{-3}$, we shall obtain

$$\chi(x) = x\left(A_0 + A_1\frac{x}{2^3} + A_2\frac{x^2}{3^3} + \ldots\right).$$

But
$$\int_0^1 v^{n-1}\left(\log_\epsilon \frac{1}{v}\right)^2 dv = \frac{1}{n^3}\,\Gamma(3)$$

Therefore we shall find

$$\chi(x) = \frac{x}{2}\int_0^1 dv\left(\log_\epsilon \frac{1}{v}\right)^2 (A_0 + A_1 vx + \ldots)$$

or
$$\int_0^1 dv\left(\log_\epsilon \frac{1}{v}\right)^2 \phi\,(vx) = \frac{2\chi(x)}{x}.$$

As $x\phi\,(x)$ or $\left(x\,\dfrac{d}{dx}\right)^3 \chi\,(x)$ can have no constant term, all the terms of the expanded form of $\left(x\,\dfrac{d}{dx}\right)^{-3} x\phi\,(x)$ are suitable for the application of the definite integral.

Again let
$$\phi(x) = \left(x^2\frac{d^3}{dx^3} + 9x\,\frac{d^2}{dx^2} + 15\,\frac{d}{dx}\right)\lambda(x)$$

then
$$x\phi(x) = \left(x^{-3}\frac{d}{dx}x^3\frac{d}{dx}x^3\frac{d}{dx}\right)\chi(x)$$

so if
$$\chi(x) = \frac{d}{dx}^{-1} x^{-3}\,\frac{d}{dx}^{-1} x^{-3}\,\frac{d}{dx}^{-1} x^3\;.\;x\phi(x)$$

and
$$x\phi\,(x) = A_0 x + A_1 x^2 + \ldots + A_{n-1} - x^n + \ldots$$

we find
$$\chi(x) = \frac{A_0 x}{1\,.\,3\,.\,5} + \frac{A_1 x^2}{2\,.\,4\,.\,6} + \cdots + \frac{A_{n-1}x^n}{n(n+2)(n+4)} + \cdots$$

$$= \frac{1}{2^3}\left\{\frac{A_0\Gamma\frac{1}{2}\,.\,x}{\Gamma\frac{7}{2}} + \frac{A_1 x^2\Gamma(1)}{\Gamma(4)}\cdots + \frac{A_{n-1}x^n\Gamma\frac{n}{2}}{\Gamma(\frac{n}{2}+3)} + \cdots\right\}$$

$$= \frac{1}{2^4}\left\{\frac{A_0\Gamma\frac{1}{2}\Gamma(3)}{\Gamma\frac{7}{2}}x + \cdots + \frac{A_{n-1}\Gamma\frac{n}{2}\Gamma 3}{\Gamma(\frac{n}{2}+3)}\,x^n + \right\}$$

$$= \frac{x}{2^4}\left\{\int_0^1 \frac{dv}{\sqrt{v}}\,(1-v)^2(A_0 + A_1 x\sqrt{v} + \ldots)\right\}$$

$$= \frac{x}{2^4}\int_0^1 \frac{dv}{\sqrt{v}}\,(1-v)^2\phi\,(x\sqrt{v})$$

and so
$$\int_0^1 \frac{dv}{\sqrt{v}}\,(1-v)^2\phi\,(x\sqrt{v}) = \frac{2^4\chi(x)}{x}$$

or if we please
$$\int_0^1 du\,(1-u^2)^2\phi\,(xu) = \frac{2^3\chi(x)}{x}.$$

If we put $\chi(x) = x^2 + x$ in this integral, we shall obtain a perfectly correct result.

I discovered the following integral some years ago. It may have been discovered before, although I have been unable to meet with it.

$$\int_0^{\frac{n}{2}} d\theta\, \theta\, (2\cos\theta)^{m-1} \sin\,(m + 2r + 1)\,\theta$$

$$= \pm\,\frac{n}{4} \cdot \frac{1\,.\,2\,.\,3\,\ldots\,.\,r}{m\,(m+1)\,(m+2)\ldots.\,(m+r)}\,.$$

From this may be deduced an enormous number of results, as will be at once apparent. I will write down two of them.

$$\int_0^{\frac{n}{2}} d\theta\,\theta\,\frac{\cos 5\theta \sin\theta + (1 - x)\sin 5\theta\cos\theta}{x^2 + 2x + 2 + (x^2 + 2x)\cos 2\theta}$$

$$= n\,\left\{\frac{(x+2)^2}{4x^3}\log_e\left(1 + \frac{x}{2}\right) - \frac{3x + 4}{8x^2}\right\}.$$

Now let $\qquad \Theta_r = \cos^{n-r}\theta \sin\,(r + 2)\,\theta.$

Then $\qquad \displaystyle\int_0^{\frac{n}{2}} \theta\,\frac{\Theta_4 - 4\alpha\Theta_3 + 6x^2\Theta_2 - 4\alpha_3\Theta_1 + \alpha^4\Theta_0}{((\alpha^2 - 2\alpha + 2) + (\alpha^2 - 2\alpha)\cos 2\theta)^4}\,d\theta$

$$= \frac{\pi}{24} \cdot \frac{1}{8 - 4\alpha}\,.$$

The first integral was derived from the series $\dfrac{x^3}{1\,.\,2\,.\,3} - \dfrac{x^4}{2\,.\,3\,.\,4} + \ldots$, the second from $\dfrac{1\,.\,2\,.\,3}{1\,.\,2\,.\,3} + \dfrac{2\,.\,3\,.\,4}{2\,.\,3\,.\,4}\,\alpha + \ldots$

VI. "On Meldrum's Rules for handling Ships in the Southern Indian Ocean." By Hon. Ralph Abercromby, F.R. Met. Soc. Communicated by R. H. Scott, F.R.S. Received June 7, 1888.

(Abstract.)

The results of this paper may be summarised as follows:—

The author examines critically certain rules given by Mr. C. Meldrum for handling ships during hurricanes in the South Indian Ocean, by means both of published observations and from personal inspection of many unpublished records in the Observatory at Mauritius. The result confirms the value of Mr. Meldrum's rules; and the author then develops certain explanations, which have been partially given by Meldrum, adds slightly to the rules for handling