In the case of liquids a quadrant electrometer was immersed in the liquid in question, and the deflection observed at different temperatures. The liquid was heated in a water-bath, and the needle and quadrants were attached to insulating supports above the bath. The electromotive force was obtained from a Ruhmkorff coil without the condenser, and with a high resistance between the terminals by which to control the E.M.F. The poles of a second electrometer in air were connected to the poles of the liquid electrometer, and the ratio of the readings of these two gave a measure of the specific inductive capacity independent of variations of E.M.F. The results are shown in the following table; and in the last column are inserted for comparison the rate of change of refractive index for the four of the liquids for which Messrs. Dale and Gladstone have determined it. Mean values are given except for these four. For glycerine there is no similarity between the two effects; but for the other three the effects are of the same order of magnitude, although not exactly in the ratio 1 : 2 indicated by the electromagnetic theory of light.

	Rate of decrease of specific inductive capacity per degree.		Rate of decrease of refractive index per degree for A line in solar spectrum.	
Turpentine......	between 20° and 36°	0·0012	between 10° and 47°	0·00035
"	" " 49	0·0011		
"	" " 62	0·0009		
Carbon bisulphide.	" 15 and 43	0·004		
Glycerine	" 18 and 41	0·006	between 20° and 48°	0·00018
"	" " 61	0·0053		
Benzoline	" 19 and 41	0·0006	between 25° and 39°	0·00037
"	" " 52	0·0011		
"	" " 63	0·0015		
Benzine	" 15 and 39	0·0014	between 10° and 39°	0·0004
"	" " 58·5	0·0012		
Olive oil	" 17 and 68	0·0024		
Paraffin oil......	" 18 and 54	increase 0·0023		

XII. "On the Interchange of the Variables in certain Linear Differential Operators." By E. B. Elliott, M.A., Fellow of Queen's College, Oxford. Communicated by Professor Sylvester, F.R.S. Received June 5, 1889.

(Abstract.)

Recent theories of functional differential invariants, reciprocants, cyclicants, &c., have brought into notice a considerable number of

linear differential operators, whose arguments are the derivatives of one with regard to the others of a set of variables connected by any single relation. By aid of such operators, in their quality of annihilators or generators, the forms of classes of functions of the derivatives having properties of persistence in form after various classes of transformations have been discussed with some completeness, and great light has been thrown on the properties of other functions in connexion with such transformations. Often, however, in cases where the transformations dealt with have not been symmetrical in all the variables, the investigation has presupposed that a certain one, or one of a restricted set, of the variables has been chosen as the dependent variable. A complete theory of the interchange of the variables in the classes of functions has been a desideratum, and towards the attainment of that end a theory of the interchange of the variables in the operators has been a first requirement. Such a theory it is the aim of the present memoir to supply for the cases of two and of three variables. I speak of the operators appertaining to the two classes of cases as binary and ternary operators respectively. For the binary operators dealt with I adopt a general form, which is a slight extension of one introduced in an able investigation of Major MacMahon's; and for the ternary operators one that is closely analogous.

I. *Binary Operators.*

By x and y are denoted two variables connected by any relation. By x_r and y_r are meant $\frac{1}{r\,!}\frac{d^rx}{dy^r}$ and $\frac{1}{r\,!}\frac{d^ry}{dx^r}$ respectively. Let ξ and η be corresponding finite increments of x and y, so that

$$\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \,\ldots\ldots,$$

and consequently

$$\begin{aligned}\xi^m &= (x_1\eta + x_2\eta^2 + x_3\eta^3 + \,\ldots\ldots)^m \\ &= \mathrm{X}_m^{(m)}\eta^m + \mathrm{X}_{m+1}^{(m)}\,\eta^{m+1} + \mathrm{X}_{m+2}^{(m)}\,\eta^{m+2} + \,\ldots\ldots, \text{ say.}\end{aligned}$$

In like manner let $\mathrm{Y}_m^{(m)}$, $\mathrm{Y}_{m+1}^{(m)}$, $\mathrm{Y}_{m+2}^{(m)}$, ... be defined.

Denote the operator

$$\frac{1}{m}\Sigma\left\{(\mu+\nu s)\mathrm{X}_s^{(m)}\frac{d}{dx_{n+s}}\right\} \text{ by } \{\mu,\,\nu\,;\;m,\,n\}_x,$$

the summation being with regard to s, which assumes in turn all integral values not less than the least of m and $-n+1$. Fractional values of m and n are not admissible, but their integral values may

be either positive or negative. The value zero of n is admitted, and that of m, though somewhat special, is not excluded.

What is sought and effected is the expression of any such operator $\{\mu, \nu; m, n\}_x$ in terms of operators of the same form $\{\mu', \nu'; m', n'\}_y$, in y dependent. The process depends on the use of a certain symbolical form for $\{\mu, \nu; m, n\}_x$, and on the proof that a simple factor produces from that symbolical form the symbolical form of the equivalent y operator.

If $m+n \geqq 1$, so that none of the coefficients of powers of η in ξ^m is wanting from $\{\mu, \nu; m, n\}_x$, the inclusive formula of transformation is found to be—

$$\{\mu, \nu;\ m, n\}_x = -\left\{\nu(n+1), \frac{\mu}{m};\ n+1,\ m-1\right\}_y;$$

and the conclusion is deduced, among others, that there are two classes of self reciprocal operators, a class of positive and one of negative character, viz. :—

$$\{-m, 1;\ m, m-1\}_x = \{-m, 1;\ m, m-1\}_y,$$

and

$$\{m, 1;\ m, m-1\}_x = -\{m, 1;\ m, m-1\}_y.$$

Particular attention is devoted to the special cases of $m = 0$ and $n = -1$; also to the transformation of Ω and V, the annihilators of invariants and of pure reciprocants. V of course, not involving the first derivative, is not an operator of the class $\{\mu, \nu;\ m, n\}$ itself, but is linear in such operators.

If $m = -n$ the formula of transformation is found to be

$$\begin{aligned} &m\{\mu, \nu;\ m, -m\}_x \\ &\quad = -\{\nu m(1-m), \mu;\ 1-m, m-1\}_y + (\mu+m\nu)y_1^{-m}\{0, 1;\ 1, -1\}_y. \end{aligned}$$

In particular

$$m\left\{\mu, -\frac{\mu}{m};\ m, -m\right\}_x = (1-m)\left\{\mu, -\frac{\mu}{1-m};\ 1-m,\ m-1\right\}_y.$$

If $m+n<0, = -r$ say, it is

$$\begin{aligned} m\{\mu, \nu;\ m, -m-r\}_x = &-\{\nu m(1-m-r), \mu;\ 1-m-r, m-1\}_y \\ &+(\mu+\nu m)X_m^{(m)}\{0, 1;\ 1-r, -1\}_y \\ &+(\mu+\nu m+\nu)X_{m+1}^{(m)}\{0, 1;\ 2-r, -1\} \\ &\qquad\qquad\qquad + \dots\dots \\ &+(\mu+\nu m+\nu r-\nu)X_{m+r-1}^{(m)}\{0, 1;\ 0, -1\} \\ &+(\mu+\nu m+\nu r)X_{m+r}^{(m)}\{0, 1;\ 1, -1\}_y. \end{aligned}$$

II. *Ternary Operators.*

Let x, y, z be variables connected by a relation of any form known or unknown. Let x_{rs}, y_{rs}, z_{rs} denote respectively

$$\frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}x}{dy^r dz^s},\ \frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}y}{dz^r dx^s},\ \frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}z}{dx^r dy^s}.$$

Let
$$\xi^m = \left\{\sum_{p+q<1} x_{pq}\eta^p\zeta^q\right\}^m = \sum_{r+s<m} X_{rs}^{(m)}\eta^r\zeta^s$$

be the expansion of the mth power of an increment of x in terms of the corresponding increments of y and z; and define as

$$m\{\mu,\ \nu,\ \nu'\,;\ m,\ n,\ n'\}_x$$

the operator
$$\Sigma(\mu+\nu r+\nu' s)X_{rs}^{(m)}\,\frac{d}{dx_{n+r,\,n'+s}},$$

and as $m\{\mu,\ \nu,\ \nu'\,;\ m,\ n,\ n'\}_y$, $m\{\mu,\ \nu,\ \nu'\,;\ m,\ n,\ n'\}_z$, the operators obtained from this by cyclically interchanging x, y, z once and twice respectively.

Attention is confined to positive integral values of m, except that the value zero of m is admitted, in so far as its admission requires the introduction of no new idea. By n and n' are denoted positive integers or zeroes, or in certain special cases -1. Thus the field of investigation is narrower than the analogue of that covered in dealing with binary operators.

The comprehensive theorem for the transformation of these ternary operators is that

$$\{\mu,\ \nu,\ \nu'\,;\ m,\ n,\ n'\}_x = \left\{-\nu(n+1),\ \nu',\ -\frac{\mu}{m}\,;\ n+1,\ n',\ m-1\right\}_y$$
$$= \left\{-\nu'(n'+1),\ -\frac{\mu}{m},\ \nu\,;\ n'+1,\ m-1,\ n\right\}_z.$$

There are three classes of cyclically persistent operators, of different characters, each corresponding to a cube root of unity, viz. :—

$$\{-m,\ 1,\ 1\,;\ m,\ m-1,\ m-1\}_x = \{-m,\ 1,\ 1\,;\ m,\ m-1,\ m-1\}_y$$
$$= \{-m,\ 1,\ 1\,;\ m,\ m-1,\ m-1\}_z,$$

$$\{-m,\ \omega,\ \omega^2\,;\ m,\ m-1,\ m-1\}_x = \omega\{-m,\ \omega,\ \omega^2\,;\ m,\ m-1,\ m-1\}_y$$
$$= \omega^2\{-m,\ \omega,\ \omega^2\,;\ m,\ m-1,\ m-1\}_z,$$

$$\{-m,\ \omega^2,\ \omega\,;\ m,\ m-1,\ m-1\}_x = \omega^2\{-m,\ \omega^2,\ \omega\,;\ m,\ m-1,\ m-1\}_y$$
$$= \omega\{-m,\ \omega^2,\ \omega\,;\ m,\ m-1,\ m-1\}_z.$$

Most of the ternary operators which in recent investigations have had their importance established, do not involve first derivatives. They are the results of replacing first derivatives by zeroes in operators such as above, or may be regarded as linear functions of different operators. The transformation of the various annihilators of pure and projective cyclicants is considered from the latter point of view.

It is indicated, however, without much development that, if preferred, it is possible to consider the transformation of operators free from first derivatives without use of operators in which those derivatives occur. In illustration of the method it is established that, if $[\mu, \nu, \nu'; m, n, n']$ denote that part of $\{\mu, \nu, \nu'; m, n, n'\}$ which is free from first derivatives,

$$(x_{10}x_{01})^{\frac{1}{3}(1-m)}[-m, 1, 1; m, 0, 0]_x = (y_{10}y_{01})^{\frac{1}{3}(1-m)}[-m, 1, 1; m, 0, 0]_y$$
$$= (z_{10}z_{01})^{\frac{1}{3}(1-m)}[-m, 1, 1; m, 0, 0]_z$$

gives for different values of m a class of cyclically persistent operators.

XIII. "On the Rate of Decomposition of Chlorine-water by Light." By G. Gore, LL.D., F.R.S. Received June 13, 1889.

(Abstract.)

In this research, the author has investigated by means of the voltaic balance the kind and amount of chemical change, the rate at which decomposition proceeds, and the chemical composition of the products formed at all stages of decomposition of chlorine-water, when exposed to daylight and sunlight in colourless glass vessels.

The chlorine-water, by exposure to diffused daylight, was decomposed with moderate uniformity, but at a gradually diminishing rate, as shown by the losses of voltaic energy, until no further loss of such energy occurred; the liquid then consisted of an aqueous solution of hydrochloric acid, hypochlorous acid, and chloric acid. By further exposure of the liquid to daylight and sunlight during several weeks, peroxide of hydrogen was formed; and the amount of hydrochloric acid and of voltaic energy very slowly increased until that of the latter became about equal to that of dilute hydrochloric acid of equivalent strength to the whole of the chlorine present; all the other chief properties of the final liquid agreed with those of a mixture of dilute hydrochloric acid and peroxide of hydrogen. Still further exposure to strong sunlight caused no further change in chemical composition, amount of voltaic energy, or other property of the liquid.