From these forms for G_n^σ and $G_n^\sigma I_n^\sigma$ a variety of others may be obtained, as well as expressions in the form of integrals for spherical harmonics of the second kind. Corresponding forms may also be established for oblate spheroids.

7. The expansion of the reciprocal of the distance between two points plays an important part in the application of these investigations. It has therefore been found in ellipsoidal harmonics and thence, by reduction, in harmonics of the spheroid, circular cylinder, and paraboloid of revolution, and its application has been briefly illustrated in finding the general term in the expansion of the potential due to the magnetism induced in an ellipsoid placed in any field of force, and in finding the electrical capacities of surfaces inverted from ellipsoids. In the same connexion, I have also found the expansion for the potential due to a thin shell bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point.

8. In the last part of the paper I have shown how to prove what Heine terms "addition theorems" in the case of spheroidal harmonics, and thence, by reduction, in the case of Bessel's functions.

II. "Photometric Observations of the Sun and Sky." By William Brennand. Communicated by C. B. Clarke, F.R.S. Received October 30, 1890.

(Abstract.)

1. The paper begins with a short account of the various papers communicated by Sir H. Roscoe, and published in the Transactions of the Royal Society.

2. My observations were made at Dacca, East Bengal, in 1861–66, repeated at Milverton, in Somersetshire, during the last two years. My first experiments were directed to ascertaining the action of the sun on sensitised paper exposed at right angles to the solar rays for different altitudes of the sun, and largely to ascertaining the laws of distribution of the actinic power in the sky.

I take no observations except when the sky is quite clear.

3. The method of measurement I adopted is the darkening produced in sensitised paper. I cut strips from one uniform sheet of ordinary photographic paper. My observations being relative, I obtain the same results (ratios) with any paper. I compare ultimately the effects of the sun and of a candle on this same paper.

4. I assume that, in burning a stearine candle, the chemical action is proportional to the material consumed; I have taken as my unit (i)

of measure of chemical action the darkening produced at a distance of 1 inch from the wick of the candle when 100 grains were consumed, which in the candle I used in India occupied about forty-seven minutes. My observations, being almost entirely relative, are independent of these assumptions, which affect hardly any of my results except comparisons with the absolute unit measures of Sir H. Roscoe.

5. Explains the method by which I obtain a standard strip for the candle unit.

7. Describes the water motion actinometer, with which observations of the action of sun and sky were made.

8. Shows how it may be proved experimentally that the intensity of the action of light emanating from a physical point varies inversely as the square of the distance from the origin.

9. For obtaining the effects of the sun and sky, I have always experimented mainly by exposing the paper at right angles to the sun's rays. Sir H. Roscoe, on the other hand, exposes his paper on a horizontal plane. Theoretic considerations have led me to another method of observation (with the "octant" actinometer below) which gives directly the measure of the effect really desired.

A table is given of the first observations I made, which afterwards led to the formation of Table B (see next page).

11. The method of observing the action of the sun alone.

12. Observations taken near the horizon not to be depended upon.

13. Refers to the construction of Table B, and the extension of the table for altitudes of the sun beyond those observed.

14. Shows how the numbers of the table were obtained, by taking the inverse of the times required at each altitude for producing the darkening of the candle unit.

17. I found the chemical action of the sun, as far as my experiments went, the same at all hours of the day and at all seasons of the year. And in Somersetshire I got exactly the same chemical action of the sun as at Dacca.

18. Various observations had led me to suspect that the chemical action of the sky at the same moment was diverse in different parts of it. To investigate this suspicion, I designed an instrument which I call the Mitrailleuse Actinometer (fig. 2). I mount a number of similar cylindrical tubes in one plane in a semicircle, to the centre of which the axis of each tube is directed: one extremity of each tube lies in the circumference of the circle; the other extremities lie on a concentric circle of about one-half the radius. In the circumference of this smaller circle is a semicircular series of holes, against which a semicircular block carrying the sensitised paper is pressed by a screw. Each cylinder cuts out of the sky a circle of 8° 28′ angular diameter. One of the tubes near its top carries a small plate of wood

Table B.

Chemical Action of Sun and Sky.

Sun's altitude.	Sun alone.	Sky alone.	Sun and sky together.	Sun's altitude.	Sun alone.	Sky alone.	Sun and sky together.
1...	0·001	0·003	0·004	31...	0·110	0·064	0·174
2...	0·002	0·005	0·006	32...	0·113	0·064	0·178
3...	0·003	0·007	0·010	33...	0·116	0·065	0·181
4...	0·004	0·010	0·014	34...	0·118	0·065	0·184
5...	0·006	0·012	0·019	35...	0·121	0·066	0·188
6...	0·009	0·016	0·025	36...	0·124	0·066	0·190
7...	0·012	0·019	0·031	37...	0·126	0·067	0·193
8...	0·016	0·022	0·038	38...	0·129	0·067	0·196
9...	0·020	0·026	0·045	39...	0·131	0·068	0·199
10...	0·024	0·029	0·052	40...	0·133	0·068	0·201
11...	0·028	0·032	0·060	41...	0·135	0·069	0·204
12...	0·033	0·035	0·068	42...	0·137	0·069	0·206
13...	0·038	0·038	0·075	43...	0·138	0·069	0·208
14...	0·043	0·040	0·082	44...	0·141	0·069	0·210
15...	0·047	0·042	0·090	45...	0·143	0·070	0·213
16...	0·052	0·045	0·097				
17...	0·057	0·048	0·105				
18...	0·061	0·049	0·110				
19...	0·066	0·050	0·116				
20...	0·070	0·052	0·122	50...	0·150	0·071	0·221
21...	0·075	0·053	0·128	55...	0·157	0·072	0·229
22...	0·079	0·054	0·134	60...	0·162	0·073	0·225
23...	0·083	0·056	0·139	65...	0·166	0·073	0·239
24...	0·086	0·057	0·144	70...	0·170	0·073	0·243
25...	0·091	0·058	0·149	75...	0·172	0·074	0·246
26...	0·094	0·059	0·153	80...	0·173	0·074	0·248
27...	0·097	0·060	0·158	85...	0·175	0·074	0·249
28...	0·101	0·061	0·169	90...	0·175	0·074	9·249
29...	0·104	0·062	0·166				
30...	0·107	0·063	0·170				

N.B.—For sun altitudes 50° to 90°, the figures are not the result of direct observations; for sun altitudes 1° to 10°, the figures are less certain by reason of thin haze often present.

on which stands a style parallel to the tube, by means of which this particular tube can be brought in a line with the sun. By another motion the plane of the tubes can be adjusted to the plane of symmetry (or elsewhere).

[A vertical plane through the sun at any time divides the visible sky into two exactly similar portions. I call this the plane of symmetry.]

19. The observations (Table C) were taken 23rd December, 1864, at Dacca (among other similar observations taken in the same cold

Table C.

(Sun's Altitude = 42° 28′.)

Altitude of the axis of the barrel of the mitrailleuse.	Distance of axis of barrel from the sun = θ.	Observed chemical action during six minutes' exposure = i_θ.	Calculated value of i_θ from i_θ = 0·12 cosec θ.
10°	−32° 58′	0·2	0·221
20	−22 58	0·5	0·308
30	−12 58	0·7	0·535
40	− 2 58	..	2·319
50	7 2	0·844	0·98
60	17 2	0·322	0·41
70	27 2	0·188	0·264
80	37 2	0·184	0·199
90	47 2	0·177	0·164
100	57 2	0·144	0·143
110	67 2	0·14	0·1304
120	77 2	0·128	0·123
130	87 2	0·122	0·12
140	97 2	0·12	0·121
150	107 2	0·128	0·126
160	117 2	0·136	0·135
170	127 2	0·136	0·156

weather) in the plane of symmetry. The barrels of the mitrailleuse were fixed 10° apart, the altitude of the sun being 42° 28′.

I give the table as an early observation that shows well that there is a point of minimum sky intensity at 90° from the sun. It also appears that if i_α be the intensity for the altitude α of the sun (= 0·12), then the intensity of the sky at a point θ° from the sun is given (roughly only according to this table) by the formula

$$i_\alpha \operatorname{cosec} \theta.$$

This observation was made in the plane of symmetry: it turns out that the value, $i_\alpha \operatorname{cosec} \theta$, gives the intensity very accurately, for any point, in any other *great* circle, whose distance from the sun is θ° measured on that circle.

20. For any altitude of the sun (α), the chemical action of the sky is a minimum at all points in a great circle, the plane of which is at right angles to the line joining its centre to the sun.

[This plane I call the plane of minimum intensity (i_α).]

As the whole of the mathematical developments of this paper are founded upon the law that at any point of the sky whose distance is θ° from the sun

$$\text{the intensity} = i_\alpha \operatorname{cosec} \theta,$$

I have been careful to verify it by numerous observations both at Dacca and in Somersetshire, and also to vary the observations in every way I could devise. Thus the mitrailleuse has been placed in the plane of minimum intensity. In this case all the barrels give the same reading for points not too near the horizon.

Next the mitrailleuse was placed in planes of great circles through the sun at various angles with the plane of symmetry, by turning it round the line joining one of its tubes with the sun the observed chemical actions agree well with

$$i_\alpha \operatorname{cosec} \theta.$$

Next by means of stops I made the aperture of each barrel of the mitrailleuse to be

$$c \sin \theta,$$

where θ° is the distance of the axis of the barrel from the sun; this mitrailleuse being exposed, the barrel $c \sin 0$ being directed to the sun, the circular darkened spots were found to be very accurately of the same depth.

Further, I calculated the *times* of exposure for a (particular) mitrailleuse with barrels of uniform aperture, which ought, on the law $i_\alpha \operatorname{cosec} \theta$, to give a uniform tint. I exposed this mitrailleuse for these calculated times, first in the plane of symmetry, afterwards in a plane inclined to it at 62°; the results agreed closely with my anticipation, and show $i_\alpha \operatorname{cosec} \theta$ to be a very good approximation.

22. I have therefore made full use of the expression $i_\alpha \operatorname{cosec} \theta$ for the chemical action of the light of the sky in a circle θ° from the sun (whose altitude is α). First, in the following proposition:—

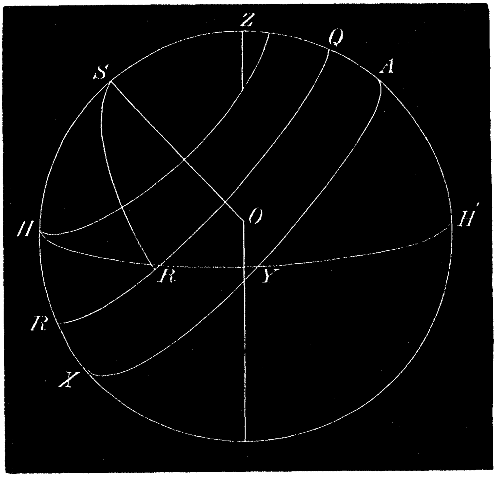
24. Having given i_α the chemical action in the circle of minimum intensity, to calculate the total chemical action of the sky on a plane exposed at right angles to the sun.

N.B.—i_α is a constant for this calculation, but it varies with the altitude of the sun.

Let the figure represent a projection on the plane of symmetry, S being the sun, Z the zenith, HRYH′ the horizon, AYX the plane of minimum intensity, SH $= \alpha$ the sun's altitude, θ the angular distance of the sun from QR. Then the total action of sky throughout the gore HYZSH on sensitised paper at O in the plane perpendicular to OS

$$= i_\alpha \left\{ \pi + 2\int_\alpha^{\frac{\pi}{2}} \frac{\tan \alpha \, d\theta}{\sqrt{(1 - \sec^2 \alpha \cos^2 \theta)}} \right\} \dots\dots\dots \text{(K.)}$$

The expression cannot be integrated; but, by using a formula of reduction in series, it gives—



Total intensity of the gore on the paper at O

$$= i_\alpha \pi \left\{ 1 + \frac{2 \sin \alpha}{1 + \sin \alpha} \left[1 + 0{\cdot}75 \tan^4 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) + 0{\cdot}14 \tan^8 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) + \&c. \right] \right\},$$

which is the formula I have used in numerical computations.

It is the numerical value in the column "Sky alone" in Table B, which is thus brought into direct verification with i_α observed by the mitrailleuse.

Arts. 25—29 show that the integral (K) taken for the whole visible hemisphere is

$$2 i_\alpha (\pi \sin \alpha + 2 \cos \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (Q).$$

This is the whole chemical action of the hemisphere resolved on the horizontal plane, which was one of the quantities observed by Sir H. Roscoe.

30. Deals with any suspicion that may arise that the law of cosecants may have been assumed, the fact being that the law was arrived at, by experiment simply, more than twenty-two years ago, &c.

31. Applies the equation (Y) to determine i_α for the altitudes given by Sir H. Roscoe in his table showing the total chemical action of diffuse daylight (*i.e.*, of the whole sky, the sun being stopped off) on horizontally exposed paper ('Phil. Trans.,' 1870, p. 314). These values are tabulated with corresponding values of i_α calculated by formula in (24) from the Dacca Table B, forming together Table E.

32. As a first approximation from Table E, it would appear that Sir H. Roscoe's unit of chemical action is $\frac{12}{16}$ of the Dacca candle unit.

Table E.

1	2	3	4	5
Sun's altitude.	Diffused daylight of Roscoe.	i_α calculated from col. 2.	i_α calculated from Dacca Table B.	Values in col. 4 brought up for comparison with those in col. 3.
9° 51′	0·038	0·0076	0·0068	0·009
19 41	0·062	0·0105	0·0107	0·0141
31 14	0·100	0·0150	0·0118	0·0156
42 13	0·115	0·0160	0·0121	0·0160
53 9	0·126	0·0170	0·0121	0·0160
61 8	0·132	0·0177	0·0120	0·0159
64 14	0·138	0·0187	0·0120	0·0159

Twilight.

33. The resultant chemical action of the sky on a horizontally exposed piece of paper, the sun's altitude being α, is found

$$= (2\pi \sin \alpha + 4 \cos \alpha) i_\alpha.$$

This vanishes when

$$2\pi \sin \alpha + 4 \cos \alpha = 0,$$

i.e., when $$\tan \alpha = -\frac{2}{\pi},$$

or $$\alpha = -32° 29'.$$

This gives an absolute value for twilight, supposing daylight to cease when the diffused daylight of Roscoe entirely vanishes.

The extreme limit at which twilight has been certainly observed is when the sun is 24° below the horizon; at which time the formula $i_\alpha(2\pi \sin \alpha + 4 \cos \alpha)$ would show the chemical action of diffuse daylight to be only $\frac{1}{40}$ of what it was just after sunset.

In other words, the formula

$$(2\pi \sin \alpha + 4 \cos \alpha) i_\alpha$$

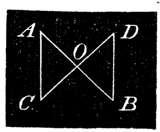
gives a very good agreement with the observed duration of twilight.

34. Taking as co-ordinate planes the plane of symmetry, the plane of minimum intensity, and the plane through the sun at right angles to these (which last I call the plane of the sun's altitude), it is found (as a corollary in Article 34) that [U], [V], [W], representing the total chemical effect of the sky, resolved on these co-ordinate planes.

This suggested the construction of the octant actinometer, which

requires only a quarter of the visible sky to be clear for observation, and gives the value of i_α directly, requiring no calculation or reduction. It possesses, moreover, the great advantage of not taking in the low band of sky near the horizon, and thus avoiding a principal element of uncertainty in other observations.

35. The octant actinometer consists of three quadrantal planes, MOS, MOI, and IOS, joined at their edges so as to form a hollow trihedral, and mounted so that one of the edges, OS, can be brought to point to the sun; the plane MOI will then coincide with the plane of minimum intensity. The instrument has another adjustment, by which it can turn round OS as an axis, and if one of the planes MOS, IOS be brought to coincide with the plane of symmetry, the other will coincide with the plane of the sun's altitude.



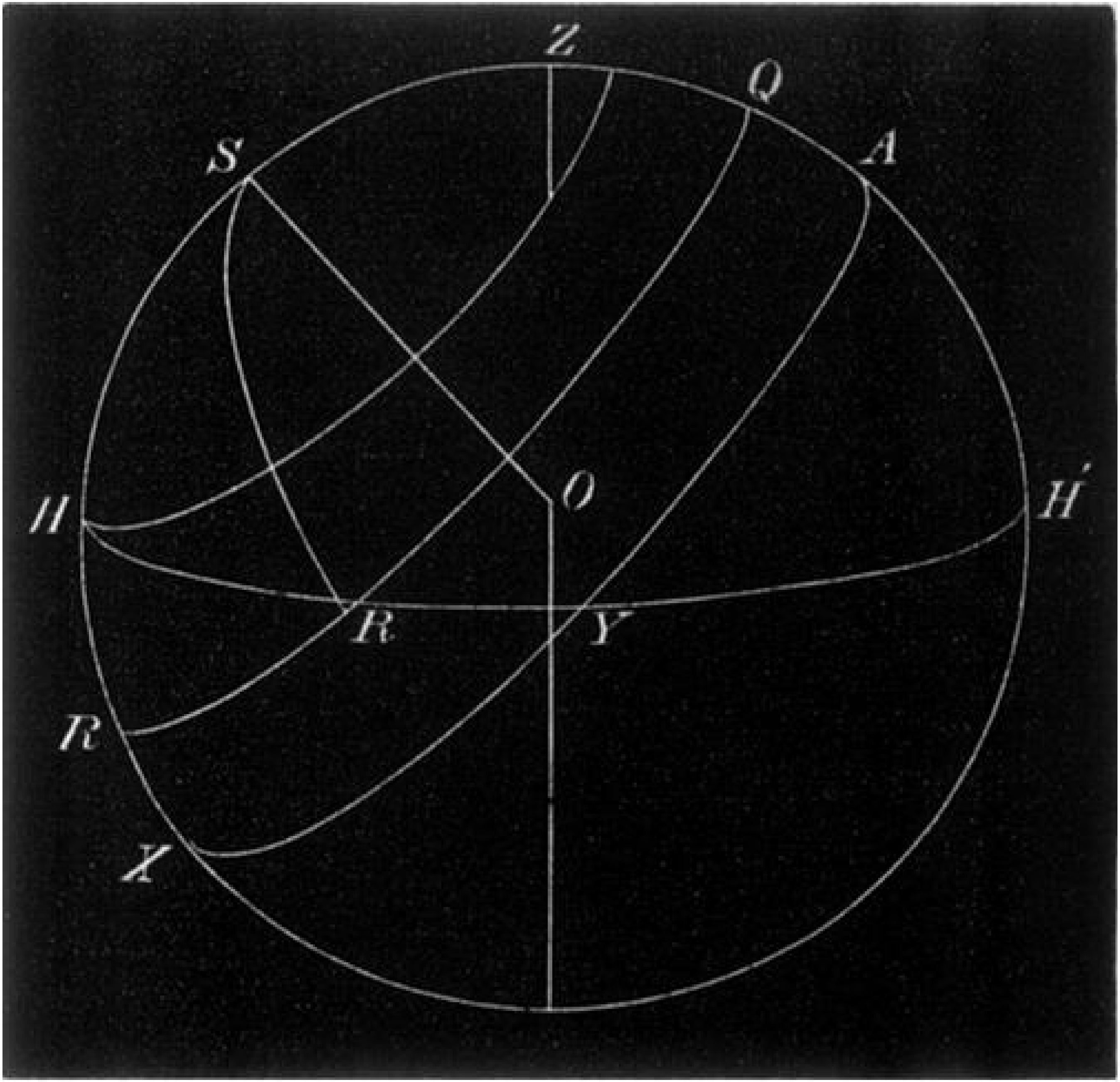
I take a small square of sensitised paper, and cut it along CO; then slipping the part COB under AOC, so that B coincides at C, it forms a rectangular trihedral of paper. This is placed in a small exposure trihedral of cardboard, and covered by a thin metal trihedral in the trihedral of the octant (I make several of these trihedrals of sensitised paper, so as in the field to take quickly a series of observations; the trihedral of paper is, of course, carefully covered till the instrument is in adjustment); exposed to the action of the sky for (say) thirty seconds, the readings on the planes MOS and IOS will be each $30 i_\alpha$, and that on the plane MOI will be $30 \cdot \frac{1}{2} \pi i_\alpha$.

36. Gives in Table F the observations with the octant in August last.

37. Discussion regarding the most useful method of resolution of the sky and sun.

III. "Determinations of the Heat Capacity and Heat of Fusion of some Substances to test the Validity of Person's Absolute Zero." By Spencer Umfreville Pickering, M.A., F.R.S. Received November 6, 1890.

The relations existing between the heat of fusion of a substance and its heat capacity in the liquid and solid condition were demonstrated by Person, in 1847 ('Ann. Chim. Phys.' (3), vol. 21, p. 315). He showed that the heat of fusion must diminish as the temperature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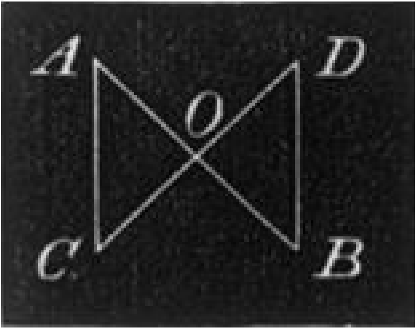
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