IV. "On the Theory of Electrodynamics." By J. LARMOR, Fellow of St. John's College, Cambridge. Communicated by Professor J. J. THOMSON, F.R.S. Received May 11, 1891.

The electrical ideas of Clerk Maxwell, which were cultivated partly in relation to mechanical models of electrodynamic action, led him to the general principle that electrical currents always flow round complete circuits.

To verify this principle for the case of the current which charges a condenser, it was necessary to postulate an electrodynamic action of the same type as that of a current for the electric displacement across the dielectric, in which the excitation of the dielectric may be supposed, after Faraday, to consist. The existence of such an action has subsequently been deduced qualitatively from the general principle of action and reaction,* and has also been detected by various experimenters.

The principle also requires that the electric displacement shall not lead to any accumulation of charge in the interior of the dielectric, therefore that it shall be solenoidal or circuital,† its characteristic equation being of the type

$$\frac{d}{dx}\left(\mathrm{K}\frac{d\mathrm{V}}{dx}\right)+\frac{d}{dy}\left(\mathrm{K}\frac{d\mathrm{V}}{dy}\right)+\frac{d}{dz}\left(\mathrm{K}\frac{d\mathrm{V}}{dz}\right)=0,$$

where V is the electric potential, and K a dielectric constant. The surface density of the electricity conducted to a face of a condenser must neutralise the electric displacement, and not leave any residual effective electrification on the surface. On taking the displacement and the surface density each equal to $\mathrm{KF}/4\pi$, where F denotes the electric force, the value of K becomes unity for a vacuum dielectric; and K represents the specific inductive capacity as measured by electrostatic experiments.

When this principle of circuital currents is postulated, the theory of electrodynamics is reduced to the Ampère-Neumann theory of complete circuits, of which the truth has been fully established. It leads, as shown by Maxwell, to the propagation of electrical action in dielectric media by waves of transverse electric displacement, which have the intimate relations to waves of light that are now well known.

* *Cf.* J. J. Thomson, 'Brit. Assoc. Report,' 1885.

† A term recently introduced by Sir W. Thomson.

*Generalised Polarisation Theory.*

The problem of determining how far these remarkable conclusions will still hold good when a more general view of the nature of dielectric polarisation is assumed was considered by von Helmholtz* in a series of memoirs.

The most general conception of the polarisation of a medium which has been formed is the Poisson theory of magnetisation. The magnetised element, whether actually produced by the orientation of polar molecules or otherwise, may be mathematically considered to be formed by the displacement of a quantity of ideal magnetic matter from its negative to its positive pole, thereby producing defect at the one end, and excess at the other end. The element is defined magnetically by its moment, which is the product of the displaced quantity and the distance through which it is displaced. The displacement per unit volume, measured by this product, is equal to the magnetic moment per unit volume, whether the magnetised molecules fill up the whole of that volume or are a system of discrete particles with unoccupied space between them.

In the electric analogue we replace ideal magnetic matter by ideal electric matter; the displacement thus measured constitutes the electric displacement, and its rate of change per unit time represents the displacement current in the dielectric. We have to consider whether a displacement current of this type suffices to make all electric currents circuital; and it will be sufficient and convenient to examine the case of a condenser which is charged through a wire connecting its two plates. In the first place this notion of electric displacement leads to the same distribution of potential between the plates as the ordinary one, adopted by Maxwell; for in the theory of induced magnetism there occurs a vector quantity of circuital character, the magnetic induction of Maxwell, of which the components are $-\mu(dV/dx)$, $-\mu(dV/dy)$, $-\mu(dV/dz)$, and which, therefore, leads to the characteristic equation of the potential

$$\frac{d}{dx}\left(\mu\frac{dV}{dx}\right)+\frac{d}{dy}\left(\mu\frac{dV}{dy}\right)+\frac{d}{dz}\left(\mu\frac{dV}{dz}\right)=0,$$

corresponding to the one given above. If the displacement in the dielectric is $-\kappa(dV/dx)$, $-\kappa(dV/dy)$, $-\kappa(dV/dz)$, then

$$\mu = 1+4\pi\kappa.$$

The displacement in a unit cube may, of course, be considered as a displacement across the opposite faces of the cube.

Now, considering the case of a plane condenser, let F be the electric force in the dielectric between the plates; then the displacement is

* 'Wissenschaftliche Abhandlungen,' I, p. 545, *et seq.*

$\kappa$F. Let $\sigma$ be the surface density of the charge conducted to a plate; then the effective electrification along that plate will be of surface density $\sigma' = \sigma - \kappa F$; therefore, by Coulomb's principle,

$$F = 4\pi\sigma'$$

$$= 4\pi(\sigma - \kappa F);$$

so that
$$\sigma = \frac{\mu}{4\pi} F = \kappa F + \frac{1}{4\pi} I$$

Thus the current is not circuital, but there is an excess of the surface density conducted to the surface over the displacement current from the surface, which is equal to $F/4\pi$.

The specific inductive capacity, as determined by static experiments on capacity, is here measured by $\mu$, the coefficient in the expression for $\sigma$.

In addition to this discontinuity at the face of a condenser plate, the induction in the mass of the dielectric will not be circuital unless the electric force is itself circuital, which it is not in the general electrodynamic theory to be presently discussed.

The current becomes more nearly circuital the greater the value of $\mu$. If $\mu$, and therefore $\kappa$, were infinite we should attain the limit when the currents are circuital. If the values of $\mu$ for all dielectrics were multiplied by the same infinite constant, so as to keep their ratios unchanged, the distribution of electric potential would not be altered, provided the charges on all conducting surfaces were also increased in that ratio; the displacement or induction, which is now the essential quantity in the theory, thus maintaining its original value. This comes to the same thing as measuring the actual charges in a unit which is diminished in that ratio.

In this way the Maxwell scheme of circuital currents reveals itself as a limiting case of the more general polarisation theory. The infinite dielectric constant makes the excited polarisation of very great amount in comparison with the exciting cause; so that in the limit we may, in a sense, imagine the system as one of self-excited circuital polarisation, a point of view which approaches somewhat to that of Maxwell himself.

This mode of connecting the two theories was pointed out by von Helmholtz. But his scheme takes for the new unit of charge the electrostatic unit corresponding to vacuum with its new infinitely great dielectric constant, so that this unit is reduced proportionally to the square root of the infinite ratio; the displacement is then infinitely great, and the potential infinitely small, according to the square root of this ratio.

(This, however, should be expressed more precisely as follows:—

The *absolute* dimensions of electric charge and electric displacement in K are both $K_2^{\frac{1}{2}}$, those of electric force (static) $K_2^{-\frac{1}{2}}$. These dimensional relations must persist when the transition is made from von Helmholtz's system to Maxwell's, so that the changes in the units are as von Helmholtz indicates; and the ratio of the electrostatic to the electrokinetic unit of quantity in an ideal absolute medium with $K_2$ unity will now be the ascertained value of this constant for air or vacuum multiplied by the square root of the value of $K_2$ for air. The electric pressure in a fluid dielectric, however, depends, in this limiting form of the theory, on the square of the value of the electric displacement, as may be proved: thus the circumstances of ordinary cases of statical electrification are those of finite numerical value of the displacement, notwithstanding the smallness of this absolute unit of charge.)

*Generalised Electrodynamic Theory.*

To obtain the general type of the modification of which the theory of electrodynamics is susceptible owing to the existence of non-circuital currents, we start, following von Helmholtz, from the ascertained laws for circuital currents, which may be developed in the manner of Neumann and Maxwell from the electrodynamic potential

$$T = \iota\iota' \iint \frac{\cos\epsilon}{r}\, ds\, ds'.$$

The value of T with the sign here given to it is to be reckoned as kinetic energy; the mechanical forces are to be derived by its variation due to any virtual displacement of the system, a force acting in the direction of the displacement producing an increment of T; the electric forces are derived according to Lenz's law or Maxwell's kinetic theory. The equations of the field are thus all expressible in terms of this function T. When non-circuital currents are contemplated, the currents $\iota$, $\iota'$, now varying with $s$, $s'$, must be placed inside the integral signs; and to T must be added the most general type of expression that will vanish when either current is circuital. Thus we must write

$$T = \iint \iota ds\, \iota' ds' \left(\frac{\cos\epsilon}{r} + \frac{d^2\Psi}{ds\, ds'}\right),$$

where $\Psi$ is a function such that

$$\int \frac{d\Psi}{ds}\, ds = 0, \qquad \int \frac{d\Psi}{ds'}\, ds' = 0,$$

*i.e.*, it is, so far, any function which has no cyclic constant round either circuit. The distribution of the energy between the pairs of

elements is now supposed to be specified by the elements of this integral.

The form of $\Psi$ is limited by the fact that it must be a function of the geometrical conformation of the pair of elements. The elements of this conformation are given by the equations

$$\cos\theta = -\frac{dr}{ds},$$

$$\cos\theta' = \frac{dr}{ds'},$$

$$\cos\epsilon = -\frac{d}{ds'}\left(r\frac{dr}{ds}\right)$$

$$= -\frac{dr}{ds}\frac{dr}{ds'} - r\frac{d^2r}{ds\,ds'},$$

where $r$ is their distance apart, $\theta, \theta', \epsilon$ represent the angles $r.ds$, $r.ds'$, $ds.ds'$, $r$ being measured positive from $ds$ to $ds'$.

The only function of the type $d^2\Psi/ds\,ds'$ which can be specified in terms of these quantities is $d^2\phi(r)/ds\,ds'$, which is equal to

$$r^{-1}\phi'(r)\,(\cos\theta\cos\theta' - \cos\epsilon) + \phi''(r)\cos\theta\cos\theta'.$$

On substitution we have

$$T = \iint \iota ds\,\iota' ds' \left\{ -\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} - \frac{d^2(r-\phi r)}{ds\,ds'} \right\},$$

in which the elements of the energy are supposed to be correctly localised.

To obtain the mutual mechanical forces between the conductors we have to determine the variation in T produced by the most general virtual displacements of the separate elements which do not alter these elements, nor break the continuity of either circuit. Thus $ds$, $ds'$, $\iota$, $\iota'$ are not to be varied.

The shortest way to take account of currents which are not of the same strength all along the circuit is to consider two uniform currents $\iota$, $\iota'$ flowing in interrupted circuits, and examine the terms of the variation involving the terminal points at which electric charges are being accumulated by the currents flowing into them. Of course the same general results would flow from taking $\iota$, $\iota'$ functions of $s$, $s'$ respectively and neglecting the ends. Thus, employing electromagnetic units and so avoiding a numerical coefficient, we have, after F. E. Neumann and von Helmholtz,

$$\mathrm{T} = \iint \iota ds\, \iota' ds' \left\{ -\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} - \frac{d^2(r-\phi r)}{ds\, ds'} \right\};$$

$$\delta\mathrm{T} = \iint \iota ds\, \iota' ds' \left\{ \frac{\delta r}{r^2}\frac{dr}{ds}\frac{dr}{ds'} - \frac{1}{r}\frac{d\delta r}{ds}\frac{dr}{ds'} - \frac{1}{r}\frac{dr}{ds}\frac{d\delta r}{ds'} - \frac{d^2\delta(r-\phi r)}{ds\, ds'} \right\}$$

$$= -\int \left| \delta r \right|_{s_1}^{s_2} \iota' ds'\, \iota \frac{1}{r}\frac{dr}{ds'} - \int \left| \delta r \right|_{s_1'}^{s_2'} \iota ds\, \iota' \frac{1}{r}\frac{dr}{ds}$$

$$+ \iint \iota ds\, \iota' ds'\, \delta r \left\{ \frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'} + \frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\right) + \frac{d}{ds'}\left(\frac{1}{r}\frac{dr}{ds}\right) \right\}$$

$$- \| \iota\iota'(1-\phi' r)\delta r \Big|_{s_2}^{s_1} \Big|_{s_1'}^{s_2'}.$$

This variation is accounted for by the following forces of repulsion, tending to increase $r$.

(i) Between the elements $\iota ds$ and $\iota' ds'$, equal to

$$\iota ds\, \iota' ds' \left( -\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'} + \frac{2}{r}\frac{d^2 r}{ds\, ds'} \right),$$

or $$-2\iota ds\, \iota' ds' \frac{1}{r^2}(\cos\epsilon - \tfrac{3}{2}\cos\theta\cos\theta'), \text{ Ampère's law.}$$

(ii) Between the element $\iota ds$ and the positive end of the conductor $ds'$,

$$\iota ds\, \iota' \frac{1}{r}\frac{dr}{ds},$$

or $$-\iota ds \frac{de'}{dt}\frac{1}{r}\cos(r\,.\,ds),$$

where $de'/dt$ is the rate at which the charge at that end is increasing.

(iii) Between the element $\iota' ds'$ and the end of the conductor $ds$,

$$\iota' ds'\, \iota \frac{1}{r}\frac{dr}{ds'},$$

or $$\iota' ds' \frac{de}{dt}\frac{1}{r}\cos(r\,.\,ds'),$$

$r$ being here measured away from $ds'$.

(iv) Between an end of one conductor and an end of another conductor,

$$-\iota\iota'(1-\phi' r),$$

or $$-\frac{de}{dt}\frac{de'}{dt}(1-\phi' r).$$

It is to be observed that the form of $\phi(r)$ affects only the forces (iv) in this scheme of attraction, as one would expect from the fact that $\phi(r)$ disappears if either current flows round a complete circuit.

Not to refer to (ii) and (iii), we notice from (iv) that two changing electrifications attract each other with a force involving a term which is constant at all distances, unless a special form of $\phi(r)$ be assigned differing from any of the values which occur in the sequel. It is difficult to imagine the mechanical basis of such an action; the remarks of von Helmholtz in justification (against Bertrand) may, however, be referred to.*

This investigation of the mechanical forces is equivalent to von Helmholtz's with the exception that he takes at the beginning $\phi(r)$ to be proportional to $r$, on the general ground that the potential energy of two elements in all natural actions involves only the inverse first power of the distance. The validity of this consideration seems to be weakened by the fact noticed above that $\phi(r)$ occurs only in the force (iv). For what follows it will not be necessary to restrict the form of $\phi(r)$.

To discuss the propagation of electrical action in continuous media, we have to translate T from the form suitable to linear distributions to the form suitable to volume distributions. Following the method first developed by Kirchhoff, and for this case the analysis of von Helmholtz, the energy function for any field of currents is to be obtained by summation of the energy functions of all the pairs of elementary filaments of currents that compose it, care being taken that no pair is counted twice over. The proper form will be a volume integral; instead of $ds$, $ds'$, the elements of the filament, it will involve $d\tau$, $d\tau'$, the elements of volume, and instead of $\iota$, $\iota'$, the resultant currents, it will involve their components per unit sectional area $uvw$ and $u'v'w'$.

Thus
$$\mathrm{T} = \iint \frac{\iota ds\, \iota' ds' \cos\epsilon}{r} + \iint \iota \frac{d}{ds}\, \iota' \frac{d}{ds'}\, \phi(r)\, ds\, ds'$$
$$= \tfrac{1}{2}\iint \frac{1}{r}\,(uu'+vv'+ww')\, d\tau\, d\tau'$$
$$+\tfrac{1}{2}\int d\tau\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)\int d\tau'\left(u'\frac{d}{dx'}+v'\frac{d}{dy'}+w'\frac{d}{dz'}\right)\phi(r),$$
the factors $\frac{1}{2}$ being inserted because the volume integrals, being extended all over the system, take each pair of elements twice over.

Hence
$$\mathrm{T} = \tfrac{1}{2}\int (\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)\, d\tau + \tfrac{1}{2}\int\left(u\frac{d\chi}{dx}+v\frac{d\chi}{dy}+w\frac{d\chi}{dz}\right)d\tau,$$

* 'Wissen. Abhandl.,' I, p. 708.

where

$$F = \int \frac{u'}{r}\, d\tau', \qquad G = \int \frac{v'}{r}\, d\tau', \qquad H = \int \frac{w'}{r}\, d\tau',$$

$$\chi = \int \left( u' \frac{d\phi}{dx'} + v' \frac{d\phi}{dy'} + w' \frac{d\phi}{dz'} \right) d\tau' ;$$

in these formulæ the accents may now be dropped, as the integrals are extended over the whole system.

It is through this function $\chi$ that the indeterminateness enters into the equations of electrodynamics. In a certain class of cases the function may be expressed in another form, which is useful in the subsequent analysis. By integration by parts throughout space, we obtain

$$\chi = \int dS\, \phi\, (lu + mv + nw) - \int d\tau\, \phi \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right)$$

$$= \int d\tau\, \phi \frac{d\rho}{dt},$$

provided we can neglect the surface integral over the infinite sphere; and this we can do, if the system is confined to a finite region and $\phi$ contains only inverse powers of $r$, or it may be direct powers of $r$ when there is no total current flow to infinity. Thus

$$\chi = -\frac{1}{4\pi} \int d\tau\, \phi\, \nabla^2 \frac{dV}{dt}$$

$$= -\frac{1}{4\pi} \int d\tau\, \frac{dV}{dt}\, \nabla^2 \phi,$$

by Green's theorem, provided the surface integrals vanish as before.

In this equation,

$$\nabla^2 \phi = \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \phi ;$$

so that, on von Helmholtz's assumption

$$\phi(r) = Cr,$$

we have

$$\nabla^2 \phi = 2C/r ;$$

and therefore

$$\chi = -\frac{1}{4\pi} \int d\tau\, \frac{2C}{r} \frac{dV}{dt}$$

so that

$$\nabla^2 \chi = 2C \frac{dV}{dt},$$

a result to be used immediately.

The final result is

$$T = \tfrac{1}{2}\int d\tau\,(F_1 u + G_1 v + H_1 w),$$

where $$F_1 = F + \frac{d\chi}{dx}, \qquad G_1 = G + \frac{d\chi}{dy}, \qquad H_1 = H + \frac{d\chi}{dz},$$

$$\chi = \int d\tau \left(u\frac{d\phi}{dx} + v\frac{d\phi}{dy} + w\frac{d\phi}{dz}\right)$$

$$= \frac{1}{4\pi}\int d\tau \frac{dV}{dt}\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right).$$

To obtain the components PQR of the electric force we assume, following F. E. Neumann and von Helmholtz, that the principle involved in Lenz's law is applicable to the element as well as to the circuit as a whole. This is the same principle as flows from Maxwell's dynamical theory, and is justified, if we assume that T is the energy function of an actual dynamical system. To the kinetic part of the electric force so determined the electrostatic part must be added, giving in all the components

$$P = -\frac{dF_1}{dt} - \frac{dV}{dx}, \qquad Q = -\frac{dG_1}{dt} - \frac{dV}{dy}, \qquad R = -\frac{dH_1}{dt} - \frac{dV}{dz}.$$

The conduction current is given by

$$\sigma\,(u_1, v_1, w_1) = (P, Q, R),$$

where $\sigma$ is the specific resistance. The total current is

$$(u, v, w) = \left(u_1 + \frac{df}{dt},\ v_1 + \frac{dg}{dt},\ w_1 + \frac{dh}{dt}\right),$$

where $$(f, g, h) = \frac{K_1}{4\pi}(P, Q, R).$$

The vector potential FGH is connected by definition above with *uvw* by the equations of potential

$$\nabla^2 (F, G, H) = -4\pi\,(u, v, w);$$

while the characteristic equation of V is

$$\nabla^2 V = -4\pi\rho$$

$$= 4\pi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right).$$

The equations of electric propagation are involved in these results.

The value of $K_1$ in electromagnetic units is very small, the square of the reciprocal of the velocity of light in the medium; so that there are, broadly, two classes of media, (i) conductors in which $K_1$ is neglected, (ii) insulators in which $u_1v_1w_1$ are zero. The equations of propagation for each case are involved in the above equations.

*Propagation in Dielectric Media.*

The simplest and most important case of this generalised theory, as displacement currents in conductors are negligible, is that of dielectrics.

In the first place, we may consider the propagation of V. We have

$$\nabla^2 V = 4\pi\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)$$

$$= -K_1\frac{d}{dt}\left(\frac{dF_1}{dx}+\frac{dG_1}{dy}+\frac{dH_1}{dz}\right)-K_1\nabla^2 V.$$

Now
$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz} = \int\frac{1}{r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)d\tau$$

$$= -\int\frac{1}{r}\frac{d\rho}{dt}\,d\tau$$

$$= -\frac{dV}{dt}.$$

Therefore
$$\frac{1+K_1}{K_1}\nabla^2 V = \frac{d^2V}{dt^2}-\frac{d}{dt}\nabla^2\chi,$$

and, finally,
$$\frac{1+K_1}{K_1}\nabla^2 V = \frac{d^2V}{dt^2}-\frac{1}{4\pi}\nabla^2\int\nabla^2\phi\,\frac{d^2V}{dt^2}\,d\tau,$$

n which $\phi$ is a function of $r$.

This equation determines the mode of propagation of V. It represents wave-motion of a complicated character which may be analysed most easily by applying the equation to the case of a plane wave with the displacement at right angles to its front. There are two comparatively simple cases.

(i.) If $\nabla^2\phi = 0$, *i.e.*, $\phi = A+Br^{-1}$, the equation becomes

$$\frac{1+K_1}{K_1}\nabla^2 V = \frac{d^2V}{dt^2}+B\frac{d^2}{dt^2}\nabla^2 V,$$

which represents wave propagation with velocity depending on the wave-length, and therefore involving dispersion.

For the plane wave

$$V \propto exp\, \iota\,(mx - nt),$$

it leads to the condition

$$\frac{1+K_1}{K_1}\, m^2 = n^2 - Bm^2n^2,$$

and the velocity of propagation is

$$v = \left(\frac{1+K_1}{K_1}\right)^{\frac{1}{2}} \left(1 - \frac{4\pi^2 B}{\lambda^2}\right)^{-\frac{1}{2}},$$

where $\lambda$ is the wave-length.

The special case of $\phi(r)$ equal to zero is worth notice, as that would represent a theory in which the element of Neumann's integral, viz., $\iota ds\, \iota' ds' \cos\epsilon/r$, is the mutual energy of two current elements. When the currents are not circuital, this leads to a condensational wave of the type here given.

(ii) If $\phi(r) = Cr$ (von Helmholtz's hypothesis) the equation becomes

$$\frac{1+K_1}{K_1}\, \nabla^2 V = (1+2C)\, \frac{d^2 V}{dt^2},$$

denoting undulatory propagation with constant velocity

$$\left\{\frac{1+K_1}{(1+2C)K_1}\right\}^{\frac{1}{2}},$$

which agrees with von Helmholtz's result, when his notation $\frac{1}{2}(k-1)$ is written for C.

There is apparently nothing self-contradictory in the more general values of $\phi(r)$. The form $\phi(r) = Cr + A + Br^{-1}$, here considered, is notable for the case $C = 1$; as then the law of electrodynamic action between two changing charges would be simply that of the inverse square.

Next we shall consider the propagation of the electric displacement $fgh$. We have

$$\nabla^2 f = \frac{K_1}{4\pi}\left(-\frac{d}{dt}\nabla^2 F - \frac{d^2}{dx\,dt}\,\nabla^2\chi - \frac{d}{dx}\,\nabla^2 V\right)$$

$$= K_1 \frac{d^2 f}{dt^2} - \frac{K_1}{4\pi}\,\nabla^2\,\frac{d^2\chi}{dx\,dt} + K_1 \frac{d\rho}{dx},$$

with two similar equations in $g$ and $h$.

From these equations $\chi$ may be eliminated by means of the equations found for V

$$\frac{1+K_1}{K_1}\nabla^2 V = \frac{d^2V}{dt^2} - \frac{d}{dt}\nabla^2\chi,$$

$$\nabla^2 V = -4\pi\rho,$$

where $$\rho = \frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}.$$

There result equations of the type

$$\left(\nabla^2 - K_1\frac{d^2}{dt^2}\right)f = \frac{K_1}{4\pi}\frac{d}{dx}\left(\frac{1+K_1}{K_1}\nabla^2 V - \frac{d^2V}{dt^2}\right) + K_1\frac{d\rho}{dx}.$$

Thus, finally, $$\left(\nabla^2 - K_1\frac{d^2}{dt^2}\right)\left(\nabla^2 f + \frac{d\rho}{dx}\right) = 0,$$

or $$\left(\nabla^2 - K_1\frac{d^2}{dt^2}\right)\left\{\frac{d}{dz}\left(\frac{df}{dz}-\frac{dh}{dx}\right) - \frac{d}{dy}\left(\frac{dg}{dx}-\frac{df}{dy}\right)\right\} = 0.$$

The three equations of this type are equivalent to only two independent equations.

They show that all displacements $fgh$ for which the condensation $\rho$ is zero are propagated with the constant velocity $K_1^{-\frac{1}{2}}$, whatever be the form assigned to $\phi(r)$. For, write

$$(f, g, h) = \left(\frac{dS}{dx}+f', \frac{dS}{dy}+g', \frac{dS}{dz}+h'\right),$$

so that $$\frac{df'}{dx}+\frac{dg'}{dy}+\frac{dh'}{dz} = 0;$$

this is possible, for to determine S we have simply

$$-\rho = \nabla^2 S,$$

so that $$S = V/4\pi.$$

These equations will then determine the mode of propagation of $f'g'h'$ subject to this condition of no condensation, because S disappears from them. The propagation of S or $V/4\pi$ has already been considered.

For a system of non-condensational waves of this kind, propagated along the axis of $x$, all the quantities must be functions of $x$; therefore $f$ must vanish; that is, the displacement must be perpendicular to the direction of propagation. These waves are therefore waves of transverse displacement.

We conclude that the propagation of waves of transverse displacement with this velocity $K_1^{-\frac{1}{2}}$ is not a characteristic of any special theory, but forms a part of any conceivable theory which admits some sort of polarisation in the dielectric, and leads to the correct results for Ampère's case of circuital currents.

This cardinal result will still follow, even if $\chi$ is any function whatever. The degree of (mathematical) generality which this remark imparts may be expressed as follows. In a complete circuit the one thing essential to the established theory is that the electric force integrated round the circuit should be equal to the time rate of change of the magnetic induction through it, and, therefore, have an ascertainable value, though its distribution round the circuit is a subject of hypothesis. The conclusion that waves of transverse displacement will be propagated in a dielectric with velocity $K_1^{-\frac{1}{2}}$ will hold good if we assume any form whatever for the electric force which does not violate this one relation, and also assume an electrostatic polarisation of the medium, equal at each point to the electric force multiplied by a constant $K_1/4\pi$. For the indeterminateness that may exist in the vector potential (or electric momentum) FGH is of the same type as that which may exist in the electric force PQR, and, therefore, as the equations show, may be merged in the latter. It would, perhaps, be difficult to conceive any more general hypothesis than this.

The increased generality which can be imparted to the theory merely leads to various modes of propagation of a condensational wave.

### *Comparison with Experimental Knowledge.*

In the general theory of polarisation sketched at the beginning of this paper,

$$(f, g, h) = \kappa\,(P, Q, R);$$

therefore

$$K_1 = 4\pi\kappa.$$

The specific inductive capacity of the medium is

$$K_2 = \mu = 1+4\pi\kappa.$$

Thus

$$K_2 = 1+K_1,$$

the units being here electrostatic.

Now, the results of various experimental investigations seem to place it beyond doubt that for dielectrics of simple chemical constitution the velocity of propagation varies as $K_2^{-\frac{1}{2}}$. Thus, in the recent experiments of Arons and Rubens,* the velocity of waves, 6 metres

* Wiedemann's 'Annalen,' vol. 42, 1891, p. 581.

long, guided by a pair of parallel wires, was measured by interference experiments when a part of the circuit was surrounded by various liquid dielectrics. The great length of the wave compared with the section of the conductor ensures that it travels with its front sensibly in the direction of propagation, and, therefore, that its velocity is normal; while the presence of the return wire limits its divergence into space. Their results are expressed in the following table which gives $K_2^{\frac{1}{2}}$, the index of refraction $m$ for light waves of length $6.10^{-7}$ metres, and the index of refraction $m'$ for the observed waves of about 6 metres long:—

| | $K_2^{\frac{1}{2}}$. | $m$. | $m'$. |
|---|---|---|---|
| Castor oil ........ | 2·16 | 1·48 | 2·05 |
| Olive oil .......... | 1·75 | 1·47 | 1·71 |
| Xylene .......... | 1·53 | 1·49 | 1·50 |
| Petroleum ........ | 1·44 | 1·45 | 1·40 |

Thus the greatest deviation from correspondence for the longer waves is about 5 per cent. The correspondence of these numbers requires that the values of $K_1$ and $K_2$ should be sensibly equal for the substances tested, which can only be the case in the limiting form of the polarisation theory which constitutes Maxwell's displacement theory. In that case, as has been seen, the currents are all circuital; the Ampère-Neumann theory of electrodynamics suffices for all purposes, and there is no condensational wave.

The stand-point from which the theory of dielectric polarisation has been generalised in the theory here expounded is that of polar elements attracting according to the law of inverse squares in the manner of small magnets. In the results, however, this conception disappears and the phenomena are all expressed in the continuous manner by means of partial differential equations.

It is also possible, in Maxwell's manner, to ignore the attractions of the elements from the beginning, and simply to define the displacement as proportional to the electric force. The statical theory of condensers shows that in the dielectric the displacement must be circuital, for the characteristic equation of the potential must hold good. The displacement constant assumed by Maxwell is equal to the specific inductive capacity, in order to ensure that the charging current shall be continuous across the faces of a condenser. It might be proposed to take a less restricted form for this constant, with the result, of course, that the currents would be non-circuital. The investigation of this paper, however, proves that in all cases the velocity of the waves of transverse displacement is specified in terms of this displacement constant; and the experimental fact that in the simpler media it is determined in the same manner by the specific inductive capacity confines us to that value of the constant which is assumed

by Maxwell.* It is necessary to emphasise that it is of the very essence of a theory of this kind that the current in the dielectric is not circuital, and, therefore, that the electric volume density produced by the electric displacement varies with the time. This is so because the electrodynamic part of the electric force is not derived from a potential. Any investigation which restricts the current to be circuital is necessarily inconsistent with itself, except for the limiting case which forms Maxwell's theory.

A discrepancy of $n$ per cent. ($n$ a small number) between the observed velocity and $K_2^{-\frac{1}{2}}$ would involve, by the formulæ at the beginning of this section, a difference of about $2n$ per cent. between $K_2$ and $K_2-1$, so that $K_2$ would be of numerical magnitude about $100/2n$; which determines the ratio in which the ordinary values of the inductive capacities of all media, including vacuum, would have to be multiplied, to make the polarisation theory not discordant with the observations.

The amount of discontinuity in the current at the surface of a conductor is the fraction $K_2^{-1}$ of the total current across the surface. At the interface between two dielectric media, denoted by the values $K_2$ and $K'_2$, the normal components of the displacement on the two sides are

$$(K_2-1)N/4\pi \quad \text{and} \quad (K'_2-1)N'/4\pi,$$

where N, N′ are the normal components of the electric force, so that

$$K_2N = K'_2N'.$$

Thus the discontinuity in the displacement is $(N'-N)/4\pi$ or $(K_2/K'_2-1)\,N/4\pi$ compared with a total displacement $(K_2-1)\,N/4\pi$; the ratio of these is $(K_2-K'_2)/K'_2(K_2-1)$, which is less than the fraction $K'^{-1}_2$, which corresponds to the surface of a conductor.

Thus, under the assumed circumstances, the ratio of the amplitudes of the condensational waves to those of the transverse waves would have a superior limit of the order $2n/100$; in the observations quoted this limit is at 5 per cent.

It is worth while to emphasise that if the polarisation theory were to take $K_2$ equal to unity for a vacuum, $K_1$ would be zero, and in a vacuum there would be nothing but action at a distance. It is thus an essential part of a theory like this that a vacuum has an absolute inductive capacity greater than unity, so that the ordinary value unity is merely a relative unit. Thus the transition to Maxwell's scheme, where the absolute coefficients are all assumed infinite, does not involve any undue stretch of the original hypothesis.

In the above, the relative velocities in different media of the

* *Cf.* J. J. Thomson, 'Brit. Assoc. Report,' 1885, p. 140.

transverse waves have been considered. The absolute velocity in a vacuum must take account of the fact that the ratio of the electrostatic and electromagnetic units of quantity has been altered by the factor $K'^{\frac{1}{2}}_2$ in the transition to Maxwell's theory, where $K'_2$ now represents the assumed absolute inductive capacity of the vacuum: thus the velocity for vacuum is $(1-K'^{-1}_2)^{-\frac{1}{2}}$ multiplied by the ratio of the electric units in vacuum, agreeing with von Helmholtz's result,* on writing this inductive capacity $K'_2$ for his constant $1+4\pi\epsilon_0$, and exceeding the velocity of light unless $K'_2$ is very great.

The theory of electrodynamics would thus appear to be, on all sides, limited to Maxwell's scheme, which has also so much to recommend it on the score of intrinsic simplicity.

The Society adjourned over the Whitsuntide Recess to Thursday, May 28.

*Presents, May* 14, 1891.

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* *Loc. cit.*, p. 627.