III. "On the Collision of Elastic Bodies." By S. H. BURBURY, F.R.S. Received October 24, 1891.

(Abstract.)

1. In this paper I discuss, firstly, a case suggested by Sir W. Thomson on June 11 as a test of the truth of the Maxwell-Boltzmann doctrine concerning the distribution of energy. Sir W. Thomson supposes a number of hollow elastic spheres, each containing a smaller sphere free to move within it. This pair he calls a doublet.

If V be the velocity of the centre of inertia of a doublet, R the relative velocity of the two spheres, then, under the distribution in question, for given direction of R, all directions of V are equally probable, and the converse is also true. If a collision occurs, the change of direction of R due to it is independent of the direction of V, as well as of the magnitudes of V and R. Therefore, after collisions, as well as before, for given direction of R all directions of V are equally probable. Whence it follows that the distribution of velocities is unaffected by collisions. This appears to me to be sound as well for internal as for external collisions.

2. The characteristic of collisions of conventional elastic bodies is the discontinuous change in the velocities without alteration of the kinetic energy. If that occurs for any material system of n degrees of freedom, there are $n-1$ independent linear functions of the velocities $v_1 \ldots . v_n$ which remain unaltered, call them $S_1 \ldots . S_{n-1}$, and one, R, which is unaltered in magnitude but reversed in sign.

3. The kinetic energy cannot contain any of the products RS, but must be of the form $2E = \lambda R^2 + f(S_1 \ldots . S_{n-1})$, where $f(S_1 \ldots . S_{n-1})$ is a quadratic function of these quantities.

4. If after collision the velocities $v'_1 \ldots . v'_n$ were all reversed in sign, R and $S_1 \ldots . S_{n-1}$ would be reversed in sign. The system would retrace its course, undergoing collision, changing v' into v. R would be positive before and negative after collision. $S_1 \ldots . S_{n-1}$ would be throughout negative, *i.e.*, of opposite signs to the signs they had in the first case.

5. To define a collision, we assume that a certain function ψ of the coordinates and constants cannot become positive, and when $\psi = 0$, $d\psi/dt$ being positive, $d\psi/dt$ changes sign discontinuously, and a collision occurs. It follows that $d\psi/dt$ is equal or proportional to R.

6. What has been proved for a system holds equally for a pair of systems, having coordinates $p_1 \ldots . p_r$ for the one, and $p_{r+1} \ldots . p_n$ for the other, if ψ be a function of $p_1 \ldots . p_n$, which cannot become positive.

7. All those systems for which, at a given instant, ψ lies between

zero and $-(d\psi/dt)\delta t$, $d\psi/dt$ being positive, will undergo collision within the time δt after that instant. Therefore $d\psi/dt$ or R measures the frequency of collision.

8. From the linear equations connecting $v_1 \ldots . v_n$ with $S_1 \ldots . S_{n-1}$ and R, we can find, say, v_1 as a linear function of $S_1 \ldots . S_{n-1}$ R, and v'_1 as the same function of $S_1 \ldots . S_{n-1}$ and $-R$. Therefore $v_1^2 - v'^2_1 = 4R\Sigma\,\mu S$, and $(v_1^2 - v'^2_1)R = 4R^2\Sigma\,\mu S$, where the μ's are functions of the coordinates and constants. Now let $S_1 \ldots . S_{n-1}$ go through all values consistent with

$$2E = \lambda R^2 + f(S_1 \ldots . S_{n-1}),$$

and let $\phi(S_1 \ldots . S_{n-1})dS_1 \ldots . dS_{n-1}$ be the number in unit volume of systems for which they lie between

$$S_1 \text{ and } S_1 + dS, \text{ \&c.},$$

given E and R and the coordinates within certain limits.

Then

$$\iiint \ldots . (v_1^2 - v'^2_1)\, R\phi\, (S_1 - S_{n-1})\, dS_1 \ldots . dS_{n-1}$$
$$= 4R^2 \iiint \ldots . \phi\, (S_1 - S_{n-1})\, \Sigma\mu S\, dS_1 \ldots . dS_{n-1}.$$

Now, in the Maxwell-Boltzmann distribution, $\phi(S_1 \ldots . S_{n-1})$ is a function of the kinetic energy only, and, therefore, constant throughout this integration. Therefore

$$\iiint \ldots . (v_1^2 - v'^2_1)\, R\, dS_1 \ldots . dS_{n-1} = 4R^2 \iint \ldots . \Sigma\,\mu S\, dS_1 \ldots . dS_{n-1}$$
$$= 0,$$

because for every set of values of $S_1 \ldots . S_{n-1}$ there is included in the integration another set with reversed signs.

Now $\iiint \ldots . (v_1^2 - v'^2_1)/R\, dS_1 \ldots . dS_{n-1}$ expresses the mean value of $v_1^2 - v'^2_1$ for all collisions, given E and R, and since it is zero, the distribution of velocities is not altered by collision, or the Maxwell-Boltzmann distribution, given existing, is not affected by collisions.

9. Certain examples are given showing the values of $S_1 \ldots . S_{n-1}$ and R in given cases, viz.:—I. Elastic spheres of masses M and m. II. Elastic spheres colliding with spheroids.

10. Professor Burnside's problem of a set of equal and similar spheres, each of which, instead of being homogeneous, has its centre of inertia at a small distance c from the centre of figure. Discussion of his result, which does not agree with the Maxwell-Boltzmann doctrine, owing, as I believe, to an oversight.

11. A general proof is now given of the permanence of the distribution, viz., if there be a set of systems called system M, each having

coordinates $p_1 \ldots\ldots p_r$, and another set called systems m with coordinates $p_{r+1} \ldots\ldots p_n$. Let $F(p_1 \ldots\ldots p_r\, v_1 \ldots\ldots v_r) dp_1 \ldots\ldots dv_r$ or $F \,.\, dp_1 \ldots\ldots dv_r$ be the number of systems M with coordinates and velocities between

$$\left.\begin{array}{c} p_1 \text{ and } p_1+dp, \\ \text{\&c.,} \\ v_1 \text{ and } v_1+dv_1, \\ \text{\&c.,} \end{array}\right\} \ldots\ldots\ldots\ldots \text{(A)},$$

and $f(p_{r+1} \ldots\ldots p_n\, v_{r+1} \ldots\ldots v_n) dp_{r+1} \ldots\ldots dv_n$ the corresponding number for the other set, between limits

$$p_{r+1} \text{ and } p_{r+1}+dp_{r+1}, \text{ \&c.} \ldots\ldots\ldots\ldots \text{(B)}.$$

Let ψ be a function, such that when $\psi = 0$ a collision occurs. Then $d\psi/dt$ or R denotes the frequency of collision. And $Ff \,.\, R \,.\, dp_1 \ldots\ldots dv_{n-1}$ denotes the number in unit of time of collisions between members of the two sets having their coordinates $p_1 \ldots\ldots p_{n-1}$, and velocities $v_1 \ldots\ldots v_n$.

Similarly, the number in unit time of collisions in the reverse direction is

$$F'f'\, R\, dp'_1 \ldots\ldots dp'_{n-1}\, dv_1 \ldots\ldots dv_{n-1}\, dR.$$

In the Maxwell-Boltzmann distribution Ff, $F'f'$ are functions of the kinetic energy only, and this being the same in the two states, $Ff = F'f'$. And as many direct as reverse collisions take place in unit time, which insures the permanence of the distribution.

12. If $Ff \neq F'f'$, then the number of systems of the first kind whose coordinates and velocities lie between

$$\begin{array}{c} p_1 \text{ and } p_1+dp_1, \\ \text{\&c.,} \\ v_1 \text{ and } v_1+dv_1, \\ \text{\&c.,} \end{array}$$

is increased per unit of time by collisions with the second set, having coordinates and velocities between

$$\begin{array}{c} p_{r+1} \text{ and } p_{r+1}+dp_{r+1}, \\ \text{\&c.,} \\ v_{r+1} \text{ and } v_{r+1}+dv_{r+1}, \\ \text{\&c.,} \end{array}$$

by the quantity

$$dp_1 \ldots\ldots dv_r\, (F'f' - Ff)\, R\, dp_{r+1} \ldots\ldots dv_{n-1}\, dR,$$

and by collision with systems m without restriction by the quantity

$$dp_1 \ldots\ldots dv_r \iint \ldots\ldots (F'f' - Ff)\, R\, dp_{r+1} \ldots\ldots dv_{n-1}\, dR,$$

in which R is a function of $v_1 \dots\dots v_n$ and the coordinates, and the integration includes all values of $p_{r+1} \dots\dots v_{n-1}$ and R.

We will suppose now (see 13, *post*) that the number F is not increased or diminished by any means except by collision with systems m. If that be so,

$$\frac{dF}{dt} = \iint \dots\dots (F'f' - Ff)\, R\, dp_1 \dots\dots dp_{n-1}\, dv_1 \dots\dots dv_{n-1}\, dR$$

and

$$\iint \dots\dots \frac{dF}{dt} \log F\, dp_1 \dots\dots dv_r$$

$$= \iint \dots\dots (F'f' - Ff)\, R \log F\, dp_1 \dots\dots dp_{n-1}\, dv_1 \dots\dots dv_{n-1}\, dR.$$

By symmetry,

$$\iint \dots\dots \frac{df}{dt} \log f\, dp_{r+1} \dots\dots dv_n$$

$$= \iint \dots\dots (F'f' - Ff)\, R \log f\, dp_1 \dots\dots dp_{n-1}\, dv_1 \dots\dots dv_{n-1}\, dR.$$

Now if

$$H = \iint \dots\dots F(\log F - 1)\, dp_1 \dots\dots dv_r + \iint \dots\dots f(\log f - 1)\, dp_{r+1} \dots\dots dv_n,$$

$$\frac{dH}{dt} = \iint \dots\dots \frac{dF}{dt} \log F\, dp_1 \dots\dots dv_r + \iint \dots\dots \frac{df}{dt} \log f\, dp_{r+1} \dots\dots dv_n,$$

and therefore

$$\frac{dH}{dt} = \iint \dots\dots (F'f' - Ff)\, R \log (Ff)\, dp_1 \dots\dots dp_n\, dv_1 \dots\dots dv_n.$$

By symmetry, as we may interchange the accents,

$$\frac{dH}{dt} = \iint \dots\dots (Ff - F'f')\, R \log (F'f')\, dp_1 \dots\dots dp_n\, dv_1 \dots\dots dv_n,$$

and therefore

$$\frac{dH}{dt} = \tfrac{1}{2} \iint \dots\dots (F'f' - Ff)\, R \log \frac{Ff}{F'f'}\, dp_1 \dots\dots dp_n\, dv_1 \dots\dots dv_n,$$

which is necessarily negative, if not zero, and then only zero when $F'f' = Ff$, that is, when the Maxwell-Boltzmann distribution prevails.

13. It can be shown in the case of rigid elastic bodies that F is not

altered except by collisions, provided $f \log f$ and $F \log F$ become zero when any of the velocities becomes infinite.

14. The rate at which H approaches its minimum is found in the case of two sets of elastic spheres of masses M and m, whose numbers in unit volume are N and n, as follows :—Let $H = H_1 + K$, where H_1 is the minimum to which H tends. K is defined to be the *disturbance* and $\frac{1}{K}\frac{dK}{dt}$ the rate of subsidence.

Suppose that the number in unit volume of spheres M having velocities between U and $U+dU$ towards the element of volume $U^2\,dU \sin\alpha\, d\alpha\, d\beta$ is,

$$N\left(\frac{h\,.\,\overline{1+D}\,M}{\pi}\right)^{\frac{3}{2}} \epsilon^{-hM(1+D)U^2}\, U^2\,dU \sin\alpha\, d\alpha\, d\beta,$$

in which $h(1+D)$ is written for h in the usual expression for that number.

Similarly for the m spheres, $h(1+d)$ shall be written for h. It is assumed that the total energy is not altered by the disturbance, which requires that

$$\frac{N}{1+D}+\frac{n}{1+d} = N+n.$$

D and d are supposed so small that D^3 and d^3 are to be neglected.

Then we find $$K = \tfrac{3}{4}\frac{n}{N}\,\overline{N+n}\, d^2,$$

and $$\frac{dK}{dt} = -\frac{8}{\sqrt{\pi}}\frac{n}{N}(N+n)^2\frac{\sqrt{(Mm)}}{(M+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}}\, d^2,$$

where s is the sum of the radii of M and m.

Hence $$\frac{1}{K}\frac{dK}{dt} = -\frac{32}{3\sqrt{\pi}}\,\overline{N+n}\,\frac{\sqrt{(Mm)}}{(M+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}} = -c,$$

and $$K = K_0\,\epsilon^{-ct},$$

$$d = d_0\,\epsilon^{-\frac{1}{2}ct}.$$

dK/dt is proportional to the density and to the square root of the absolute temperature.