Cylindrical Elastic Wave compared with corresponding Diffusive Wave through the Operators.

76. The formulæ [e] and [E] are of a somewhat different kind, since the operand is the reciprocal of the independent variable. They are proved at once by carrying out the differentiations. Thus, for [E],

$$\mathrm{I}_0(qr)\,\frac{1}{vt} = \left(1+\frac{q^2r^2}{2^2}+\frac{q^4r^4}{2^24^2}+\ldots\right)\frac{1}{vt}$$

$$= \left\{1+\frac{\lfloor 2}{2^2}\left(\frac{r}{vt}\right)^2+\frac{\lfloor 4}{2^24^2}\left(\frac{r}{vt}\right)^4+\ldots\right\}\frac{1}{vt}$$

$$= \frac{1}{(v^2t^2-r^2)^{\frac{1}{2}}}. \tag{178}$$

So, by [C] and [E] we have

$$\frac{\pi}{2}\,\mathrm{K}_0(qr)\,q = \mathrm{I}_0(qr)\,\frac{1}{vt} = \frac{1}{(v^2t^2-r^2)^{\frac{1}{2}}}. \tag{179}$$

There is an interesting analogue to this transformation from K_0 to I_0 occurring in the theory of pure diffusion. Change the meaning of q from $d/d(vt)$ to $\{d/d(vt)\}^{\frac{1}{2}}$, that is, to its square root. Then we shall have

$$\frac{\pi}{2}\,\mathrm{K}_0(qr)\,q = \mathrm{I}_0(qr)\,\frac{1}{2vt} = \frac{\epsilon^{-r^2/4vt}}{2vt}. \tag{180}$$

The quantity v is no longer a velocity, however. In the theory of heat diffusion it is the ratio of the conductivity to the capacity. This example belongs to cylindrical diffusion, and is only put here to compare with the preceding example, which belongs to the corresponding problem with elastic waves without local dissipation.

IX. "On a Failure of the Law in Photography that when the Products of the Intensity of the Light acting and of the Time of Exposure are Equal, Equal Amounts of Chemical Action will be produced." By Captain W. DE W. ABNEY, C.B., F.R.S. Received June 13, 1893.

It has been generally assumed that when the products of the intensity of light acting on a sensitive surface and the time of exposure are equal similar amounts of chemical action are produced, and with the ordinary exposures and intensities of light employed such, no doubt, is practically the case, and any methods of measurement hitherto practicable have been insufficiently delicate to discover any departure from this law, if such departure existed. In some recent experiments

however, I have discovered that this law breaks down under certain conditions, and I think the fact worthy the attention of those interested in the subject, since it is possible that these conditions may arise with other experimenters. Quite lately I have described the method of comparing the photographic value of sunlight with that of candle light ('Photographic Journal,' June, 1893), which was as follows:—A beam of sunlight, after three reflections from plain glass mirrors, was admitted through a narrow slit to sensitive bromide paper stretched round a drum of about 4 inches in diameter. The drum could be caused to rotate round its axis at any speed up to about sixty revolutions per second, by means of an electro-motor. A small exposure with this light was given to the paper during the rotation of the cylinder. Subsequently an amyl acetate lamp was placed in position at any convenient distance from the same slit, and a fresh portion of the same sensitive paper exposed to its action during a much longer period, the rotation being continued as before. The slit was next replaced by a small square aperture, of some $\frac{1}{2}$ inch side, and further portions of the same paper exposed to the amyl acetate light at the same distance, for varying but known exposures, with the drum at rest. On development the paper showed three images, a narrow band of deposit of the width of the slit caused by the sunlight, a second band of the same width due to the light from the amyl acetate lamp, and a third row of squares of varying blackness of deposit due to the different exposures given with the drum at rest.

If the width of the slit be accurately measured, the band formed by the amyl acetate lamp is evidently superfluous, supposing the usually accepted law to hold good under all circumstances, as by measuring the blackness, or rather want of whiteness, of the different squares, and using them as ordinates to the abscissæ which were the times of exposure, and drawing a curve through them, the blackness produced by the sunlight could be referred to that produced by the light of the amyl acetate lamp, and its equivalent value in terms of the latter light be calculated. The band of deposit produced by the amyl acetate lamp was introduced as a check, for its blackness could also be referred to the curve, and the width of the slit be calculated from it. On making such calculations I was surprised to find that in every case the calculated width of the slit was always considerably less than what it was in reality, the difference being far beyond that which would be caused by any error in the measurement. This led me to commence an investigation into the cause of this difference, and what has already been carried out is sufficient to show that there is a failure in the usually accepted law. It may be pointed out that if it held good the sum of any number of very short exposures should be equivalent to a single exposure for the same length of time.

The experiment which naturally suggested itself was to expose a sensitive surface to the action of the light of an amyl acetate lamp passing through a slit as before, the drum on which it was stretched being caused to rotate at high and low speeds, and also to place on the same paper a scale of exposures with the drum at rest. These were all developed together. An example of one of many experiments is given as an illustration.

The circumference of the drum with the paper stretched round it was 12·25 in. The width of the slit was arranged to be 0·012 in. The amyl acetate lamp was placed 2 ft. from the slit, and a rotation of 30 per sec. was given to the drum for one exposure and 1 per sec. for a second exposure. In the first case the time of exposure during each revolution was $\frac{0 \cdot 012}{12 \cdot 25} \times \frac{1}{30}$ sec., or about 1/30,000 sec.

The sum of the exposures during 20 min. was thus 1·176 sec.

In the other case the exposure was

$$\frac{0 \cdot 012}{12 \cdot 25}, \text{ or about } 1/1000 \text{ sec.,}$$

and the sum of the exposures was, as before, 1·176 sec. Thus the first individual exposures had only $\frac{1}{30}$ of the duration of the second exposures, though in the aggregate they were the same.

A scale of blackness was made on the same paper, through a square aperture, without shifting the lamp, the exposures being $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, and 8 sec. On developing it was apparent to the eye that the first band was much lighter than the second. The scale and blackness of the bands were measured accurately, and the times of exposure which had been given to each band, on the assumption that the law enunciated held good, were calculated and found to be for the first band 0·6 sec., and for the second band 0·91 sec., instead of 1·176 sec. which was really given in all. Another example is where the slit was opened to 0·11 in., and the time of exposure reduced from 20 to 10 min. It was found that in this case the exposures given on the same assumption were 3·7 sec. and 5·28 sec., the real exposure given being 5·36 sec. The last experiment shows that if the slit had been slightly wider or the rotation slower the law would have been approximately obeyed.

Another experiment was made by throwing an image of the crater of the positive pole of the electric light on a hole bored in a plate about $\frac{1}{20}$ in. in diameter by means of a lens, and allowing the emergent beam to fall on the slit and paper, the drum being made to rotate as before. The same kind of results were obtained.

As it might be thought that this difference was caused by some action other than chemical, another series of experiments was undertaken. In these different sensitive surfaces were employed in order

to eliminate any possibility of the effect being due to any phosphorescence of the paper, though none could be detected. Plates were held stationary and exposures made by admitting light to portions of them through slits of known angular aperture, cut in a disc which could be rotated at any desired speed. Similar results were obtained to those already described. The quickest rotation gave the least density. It may be remarked that the more sensitive a surface is to radiation the less marked are the differences observable for the same speeds of rotation. This is what might be expected.

As an outcome of the experiments so far made, it seems that when exposures less than 1/1000 sec. are made on a sensitive surface, and the source of illumination is an amyl acetate lamp (Von Altneck's) placed 1 ft. from the sensitive surface, the law quoted *ante* fails.

The question of a very low intensity of light acting and of the sensitiveness to different spectrum colours is now occupying my attention.

Addendum. July 4, 1893.

Since the above paper was read I have made an investigation into the question as to whether the foregoing law fails when feeble intensities of light are acting, and find that it does so signally. Sensitive surfaces were exposed in a Spurge sensitometer, in which there are thirty graduations of light admitted to different parts of the surface at the same time, the intensity of light being varied by its admission through apertures of varying size. The smallest aperture used was 1/256 of the largest, and an exposure lasting 2650 sec. was given to the former, whilst 10 sec. was given to the latter. It was found that the blackness produced by the two was very different, that produced by the light passing through the small aperture corresponding to an area of 1/600 of that passing through the largest aperture, if the law held good. The light employed was a large illuminated surface, which was equal to one amyl acetate lamp placed $6\frac{1}{2}$ in. away from the surface, without passing through the sensitometer.

As some persons might doubt the accuracy of this method, a different mode of experimenting was adopted in the next series made. An amyl acetate lamp was used, and portions of a sensitive surface were exposed at different distances from it, on the assumption that the squares of the distances gave a measure of the exposures necessary in order to produce equality of chemical action. In one experiment exposures were made at distances of 2, 4, 8, 12, 16, 20, and 24 ft.; the duration of exposure at 2 ft. being 10 sec., whilst for the last it was 24 min. The intermediate exposures were calculated on the same principle, a scale of blackness was also made by exposing

other parts for different times at a fixed distance from the light. As a result it was found that, on development, the deposit was greatest when the exposure had been made at 2 ft. and diminished for each successive distance. By applying the measures of the different blacknesses obtained at the different distances to the curve obtained by the measurement of the scale of exposures, it was found that the exposure at 24 ft. ought to have been prolonged by 4·3 times to give the same blackness as that at 2 ft., the other distances giving intermediate results. If the law held good, the actual blackness of deposit at 24 ft. would have been obtained had the same exposure been given at about 50 ft. Other experiments are in progress, but it seemed advisable, without waiting for their completion, to make this addition to the paper, to show that the law fails both when short exposures and also feeble intensities of light are in question.

X. "On the Displacement of a Rigid Body in Space by Rotations. Preliminary Note." By J. J. WALKER, F.R.S.

Received May 19, 1893.

Having been led to study more particularly than, as far as I am aware, has hitherto been done the conditions of the arbitrary displacement of a rigid body in space by means of rotations only, the results arrived at in the case of the single pairs of axes seem to me of sufficient interest and completeness to warrant their being recorded.

A comparison of these results with those arrived at by Rodrigues in his classic memoir "Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace . . . ." 'Liouville,' vol. 5, 1840, at once suggesting itself, it may be proper here to recall the substance of the latter, and show how far they fall short of the object I propose to myself. The case of displacement by successive rotations round a pair of axes is discussed in § 13 (pp. 395—396), where it is shown that (p. 390), "Tout déplacement d'un système solide peut être représenté d'une infinité de manières par la succession de deux rotations de ce système autour de deux axes fixes non convergents. Le produit des sinus de ces demi-rotations multipliés par le sinus de l'angle de ces axes et par leur plus courte distance, est égal, pour tous ces *couples* d'axes *conjugués*, au produit du sinus de la demi-rotation du système autour de l'axe *central* du déplacement, multiplié par la demi-translation absolue du système."

Then (p. 396) the converse of this theorem is affirmed, viz., that "*Tout déplacement . . . peut toujours provenir, d'une infinité de manières, de la succession de deux rotations autours de deux axes non-convergents* pourvu que *le produit. . . .*"

In this conversion of the theorem above, it is strangely over-