June 15, 1893.

The LORD KELVIN, D.C.L., LL.D., President, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

Professor William Tennant Gairdner and Professor James William H. Trail were admitted into the Society.

The following Papers were read:—

I. "On the Elasticity of a Crystal according to Boscovich." By the LORD KELVIN, P.R.S. Received June 8, 1893.

§ 1. A crystal in nature is essentially a homogeneous assemblage of equal and similar molecules, which for brevity I shall call crystalline molecules. The crystalline molecule may be the smallest portion which can be taken from the substance without chemical decomposition, that is to say, it may be the group of atoms kept together by chemical affinity, which constitutes what for brevity I shall call the chemical molecule; or it may be a group of two, three, or more of these chemical molecules kept together by cohesive force. In a crystal of tartaric acid the crystalline molecule may be, and it seems to me probably is, the chemical molecule, because if a crystal of tartaric acid is dissolved and recrystallised it always remains dextro-chiral. In a crystal of chlorate of soda, as has been pointed out to me by Sir George Stokes, the crystalline molecule probably consists of a group of two or more of the chemical molecules constituting chlorate of soda, because, as found by Marbach,* crystals of the substance are some of them dextro-chiral and some of them levo-chiral; and if a crystal of either chirality is dissolved the solution shows no chirality in its action on polarised light; but if it is recrystallised the crystals are found to be some of them dextro-chiral and some of them levo-chiral, as shown both by their crystalline forms and by their action on polarised light. It is possible, however, that even in chlorate of soda the crystalline molecule may be the chemical molecule, because it may be that the chemical molecule in solution has its atoms relatively mobile enough not to remain persistently in any dextro-chiral or levo-chiral grouping, and that each individual chemical molecule settles

* 'Pogg. Ann.,' vol. 91, pp. 482—487 (1854); or 'Ann. de Chimie,' vol. 43 (55), pp. 252—255.

into either a dextro-chiral or levo-chiral configuration in the act of forming a crystal.

§ 2. Certain it is that the crystalline molecule has a chiral configuration in every crystal which shows chirality in its crystalline form or which produces right- or left-handed rotation of the plane of polarisation of light passing through it. The magnetic rotation has neither right-handed nor left-handed quality (that is to say, no chirality). This was perfectly understood by Faraday and made clear in his writings, yet even to the present day we frequently find the chiral rotation and the magnetic rotation of the plane of polarised light classed together in a manner against which Faraday's original description of his discovery of the magnetic polarisation contains ample warning.

§ 3. These questions, however, of chirality and magnetic rotation do not belong to my present subject, which is merely the forcive* required to keep a crystal homogeneously strained to any infinitesimal extent from the condition in which it rests when no force acts upon it from without. In the elements of the mathematical theory of elasticity† we find that this forcive constitutes what is called a homogeneous stress, and is specified completely by six generalised force-components, $p_1, p_2, p_3, \ldots, p_6$, which are related to six corresponding generalised components of strain, $s_1, s_2, s_3, \ldots, s_6$, by the following formulas :—

$$w = \tfrac{1}{2}(p_1 s_1 + p_2 s_2 + \ldots + p_6 s_6) \quad \ldots\ldots\ldots\ldots \quad (1),$$

where w denotes the work required per unit volume to alter any portion of the crystal from its natural unstressed and unstrained condition to any condition of infinitesimal homogeneous stress or strain :

$$p_1 = \frac{dw}{ds_1}, \ldots, p_6 = \frac{dw}{ds_6} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

where $\dfrac{d}{ds_1}, \ldots, \dfrac{d}{ds_6}$ denote differential coefficients on the supposition that w is expressed as a homogeneous quadratic function of $s_1, \ldots, s_6$:

$$s_1 = \frac{\partial w}{dp_1}, \ldots, s_6 = \frac{\partial w}{dp_6} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where $\dfrac{\partial}{dp_1}, \ldots, \dfrac{\partial}{dp_6}$ denote differential coefficients on the supposition that w is expressed as a homogeneous quadratic function of $p_1, \ldots, p_6$.

* This is a word introduced by my brother, the late Professor James Thomson, to designate any system of forces.

† 'Phil. Trans.,' April 24, 1856, reprinted in vol. iii, 'Math. and Phys. Papers' (Sir W. Thomson), pp. 84—112.

§ 4. Each crystalline molecule in reality certainly experiences forcive from some of its nearest neighbours on two sides, and probably also from next nearest neighbours and others. Whatever the mutual forcive between two mutually acting crystalline molecules is in reality, and however it is produced, whether by continuous pressure in some medium, or by action at a distance, we may ideally reduce it, according to elementary statical principles, to two forces, or to one single force and a couple in a plane perpendicular to that force. Boscovich's theory, a purely mathematical idealism, makes each crystalline molecule a single point, or a group of points, and assumes that there is a mutual force between each point of one crystalline molecule and each point of neighbouring crystalline molecules, in the line joining the two points. The very simplest Boscovichian idea of a crystal is a homogeneous group of single points. The next simplest idea is a homogeneous group of double points.

§ 5. In the present communication, I demonstrate that, if we take the very simplest Boscovichian idea of a crystal, a homogeneous group of single points, we find essentially six relations between the twenty-one coefficients in the quadratic function expressing w, whether in terms of $s_1, \ldots, s_6$ or of $p_1, \ldots, p_6$. These six relations are such that infinite resistance to change of bulk involves infinite rigidity. In the particular case of an equilateral* homogeneous assemblage with such a law of force as to give equal rigidities for all directions of shearing, these six relations give $3k = 5n$, which is the relation found by Navier and Poisson in their Boscovichian theory for isotropic elasticity in a solid. This relation was shown by Stokes to be violated by many real homogeneous isotropic substances, such, for example, as jelly and india-rubber, which oppose so great resistance to compression and so small resistance to change of shape, that we may, with but little practical error, consider them as incompressible elastic solids.

§ 6. I next demonstrate that if we take the next simplest Boscovichian idea for a crystal, a homogeneous group of double points, we can assign very simple laws of variation of the forces between the points which shall give any arbitrarily assigned value to each of the twenty-one coefficients in either of the quadratic expressions for w.

§ 7. I consider particularly the problem of assigning such values to the twenty-one coefficients of either of the quadratic formulas as shall render the solid incompressible. This is most easily done by taking w as a quadratic function of $p_1, \ldots, p_6$, and by taking one of these

* That is to say, an assemblage in which the lines from any point to three neighbours nearest to it and to one another are inclined at 60° to one another; and these neighbours are at equal distances from it. This implies that each point has twelve equidistant nearest neighbours around it, and that any tetrahedron of four nearest neighbours has for its four faces four equal equilateral triangles.

generalised stress components, say p_6, as uniform positive or negative pressure in all directions. This makes s_6 uniform compression or extension in all directions, and makes $s_1, \ldots, s_5$ five distortional components with no change of bulk. The condition that the solid shall be incompressible is then simply that the coefficients of the six terms involving p_6 are each of them zero. Thus, the expression for w becomes merely a quadratic function of the five distortional stress-components, $p_1, \ldots, p_5$, with fifteen independent coefficients: and equations (3) of § 3 above express the five distortional components as linear functions of the five stress-components with these fifteen independent coefficients.

Added July 18, 1893.

§ 8. To demonstrate the propositions of § 5, let OX, OY, OZ be three mutually perpendicular lines through any point O of a homogeneous assemblage, and let x, y, z be the coordinates of any other point P of the assemblage, in its unstrained condition. As it is a homogeneous assemblage of single points that we are now considering, there must be another point P′, whose coordinates are $-x, -y, -z$. Let $(x+\delta x, y+\delta y, z+\delta z)$ be the coordinates of the altered position of P in any condition of infinitesimal strain, specified by the six symbols e, f, g, a, b, c, according to the notation of Thomson and Tait's 'Natural Philosophy,' Vol. I, Pt. II, § 669. In this notation, e, f, g denote simple infinitesimal elongations parallel to OX, OY, OZ respectively; and a, b, c infinitesimal changes from the right angles between three pairs of planes of the substance, which, in the unstrained condition, are parallel to (XOY, XOZ), (YOZ, YOX), (ZOX, ZOY) respectively (all angles being measured in terms of the radian). The definition of a, b, c may be given, in other words, as follows, with a taken as example: a denotes the difference of component motions parallel to OY of two planes of the substance at unit distance asunder, kept parallel to YOX during the displacement; or, which is the same thing, the difference of component motions parallel to OZ of two planes at unit distance asunder kept parallel to ZOX during the displacement. To avoid the unnecessary consideration of rotational displacement, we shall suppose the displacement corresponding to the strain-component a to consist of elongation perpendicular to OX in the plane through OX bisecting YOZ, and shrinkage perpendicular to OX in the plane through OX perpendicular to that bisecting plane. This displacement gives no contribution to δx, and contributes to δy and δz respectively $\frac{1}{2}az$ and $\frac{1}{2}ay$. Hence, and dealing similarly with b and c, and taking into account the contributions of e, f, g, we find

$$\left.\begin{aligned}\delta x &= ex+\tfrac{1}{2}(bz+cy)\\ \delta y &= fy+\tfrac{1}{2}(cx+az)\\ \delta z &= gz+\tfrac{1}{2}(ay+bx)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (4).$$

§ 9. In our dynamical treatment below, the following formulas, in which powers higher than squares or products of the infinitesimal ratios $\delta x/r$, $\delta y/r$, $\delta z/r$ (r denoting OP) are neglected, will be found useful.

$$\frac{\delta r}{r} = \frac{x\delta x+y\delta y+z\delta z}{r^2}+\frac{1}{2}\frac{\delta x^2+\delta y^2++\delta z^2}{r^2}-\frac{1}{2}\left(\frac{x\delta x+y\delta y+z\delta z}{r^2}\right)^2 \quad (5).$$

Now by (4) we have

$$x\delta x+y\delta y+z\delta z = ex^2+fy^2+gz^2+ayz+bzx+cxy \quad \ldots\ (6),$$

and

$$\begin{aligned}\delta x^2+\delta y^2+\delta z^2 = {} & e^2x^2+f^2y^2+g^2z^2\\ & +\tfrac{1}{4}\left[a^2(y^2+z^2)+b^2(z^2+x^2)+c^2(x^2+y^2)\right]\\ & +\left[\tfrac{1}{2}bc+(f+g)a\right]yz+\left[\tfrac{1}{2}ca+(g+e)b\right]zx+\left[\tfrac{1}{2}ab+(e+f)c\right]xy \quad (7).\end{aligned}$$

Using (6) and (7) in (5), we find

$$\frac{\delta r}{r} = r^{-2}(ex^2+fy^2+gz^2+ayz+bzx+cxy)+Q(e, f, g, a, b, c).. \quad (8),$$

where Q denotes a quadratic function of e, f, &c., with coefficients as follows:—

$$\left.\begin{array}{lll}\text{Coefficient of } \tfrac{1}{2}e^2 & \text{is} & \dfrac{x^2}{r^2}-\dfrac{x^4}{r^4}\\[2ex] \quad ,, \quad \tfrac{1}{2}a^2 & ,, & \tfrac{1}{4}\dfrac{y^2+z^2}{r^2}-\dfrac{y^2z^2}{r^4}\\[2ex] \quad ,, \quad fg & ,, & -\dfrac{y^2z^2}{r^4}\\[2ex] \quad ,, \quad bc & ,, & \tfrac{1}{4}\dfrac{yz}{r^2}-\dfrac{x^2yz}{r^4}\\[2ex] \quad ,, \quad ea & ,, & -\dfrac{x^2yz}{r^4}\\[2ex] \quad ,, \quad eb & ,, & \tfrac{1}{2}\dfrac{zx}{r^2}-\dfrac{x^3z}{r^4}\end{array}\right\} \quad \ldots\ldots\ldots\ (9),$$

and corresponding symmetrical expressions for the other fifteen coefficients.

§ 10. Going back now to § 3, let us find w, the work per unit volume, required to alter our homogeneous assemblage from its unstrained condition to the infinitesimally strained condition specified by e, f, g, a, b, c. Let $\phi(r)$ be the work required to bring two points of the system from an infinitely great distance asunder to distance r. This is what I shall call the mutual potential energy of two points at distance r. What I shall now call the potential energy of the whole system, and denote by W, is the total work which must be done to bring all the points of it from infinite mutual distances to their actual positions in the system; so that we have

$$W = \tfrac{1}{2}\Sigma\Sigma\phi(r) \qquad \text{.................. (10)},$$

where $\Sigma\phi(r)$ denotes the sum of the values of $\phi(r)$ for the distances between any one point O, and all the others; and $\Sigma\Sigma\phi(r)$ denotes the sum of these sums with the point O taken successively at every point of the system. In this double summation $\phi(r)$ is taken twice over, whence the factor $\frac{1}{2}$ in the formula (10).

§ 11. Suppose now the law of force to be such that $\phi(r)$ vanishes for every value of r greater than $\nu\lambda$, where λ denotes the distance between any one point and its nearest neighbour, and ν any small or large numeric exceeding unity, and limited only by the condition that $\nu\lambda$ is very small in comparison with the linear dimensions of the whole assemblage. This, and the homogeneousness of our assemblage, imply that, except through a very thin surface layer of thickness $\nu\lambda$, exceedingly small in comparison with diameters of the assemblage, every point experiences the same set of balancing forces from neighbours as every other point, whether the system be in what we have called its unstrained condition or in any condition whatever of homogeneous strain. This strain is not of necessity an infinitely small strain, so far as concerns the proposition just stated, although in our mathematical work we limit ourselves to strains which are infinitely small.

§ 12. Remark also that if the whole system be given as a homogeneous assemblage of any specified description, and if all points in the surface-layer be held by externally applied forces in their positions as constituents of a finite homogeneous assemblage, the whole assemblage will be in equilibrium under the influence of mutual forces between the points; because the force exerted on any point O by any point P is balanced by the equal and opposite force exerted by the point P′ at equal distance on the opposite side of O.

§ 13. Neglecting now all points in the thin surface layer, let N denote the whole number of points in the homogeneous assemblage

within it. We have, in § 10, by reason of the homogeneousness of the assemblage,

$$\Sigma\Sigma\phi(r) = \mathrm{N}\Sigma\phi(r) \ldots\ldots\ldots\ldots\ldots\ldots \quad (11),$$

and equation (10) becomes

$$\mathrm{W} = \tfrac{1}{2}\mathrm{N}\Sigma\phi(r) \ldots\ldots\ldots\ldots\ldots\ldots \quad (12).$$

Hence, by Taylor's theorem,

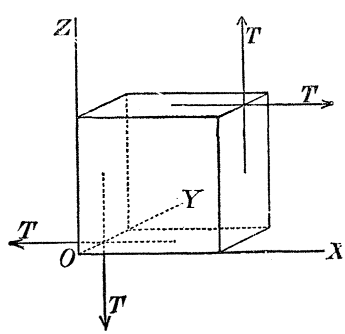
$$\delta\mathrm{W} = \tfrac{1}{2}\mathrm{N}\Sigma\{\phi'(r)\,\delta r + \tfrac{1}{2}\phi''(r)\,\delta r^2\} \ldots\ldots\ldots\ldots \quad (13);$$

and using (8) in this, and remarking that if (as in § 14 below) we take the volume of our assemblage as unity, so that N is the number of points per unit volume, δW becomes the w of § 3; we find

$$w = \tfrac{1}{2}\mathrm{N}\Sigma\left\{\frac{\phi'(r)}{r}(ex^2+fy^2+gz^2+ayz+bzx+cxy) + r\phi'(r)\,\mathrm{Q}(e, f, g, a, b, c) + \tfrac{1}{2}\frac{\phi''(r)}{r^2}(ex^2+fy^2+gz^2+ayz+bzx+cxy)^2\right\} \ldots\ldots \quad (14).$$

§ 14. Let us now suppose, for simplicity, the whole assemblage, in its unstrained condition, to be a cube of unit edge, and let P be the sum of the normal components of the extraneous forces applied to the points of the surface-layer in one of the faces of the cube. The equilibrium of the cube, as a whole, requires an equal and opposite normal component P in the opposite face of the cube. Similarly, let Q and R denote the sums of the normal components of extraneous force on the two other pairs of faces of the cube. Let T be the sum of tangential components, parallel to OZ, of the extraneous forces on either of the YZ faces. The equilibrium of the cube as a whole requires four such forces on the four faces parallel to OY, constituting

FIG. 1.



two balancing couples, as shown in the accompanying diagram. Similarly, we must have four balancing tangential forces S on the four faces parallel to OX, and four tangential forces U on the four faces parallel to OZ.

§ 15. Considering now an infinitely small change of strain in the cube from (e, f, g, a, b, c) to $(e+de, f+df, g+dg, a+da, b+db, c+dc)$; the work required to produce it, as we see by considering the definitions of the displacements e, f, g, a, b, c, explained above in § 8, is as follows,

$$dw = \mathrm{P}de + \mathrm{Q}df + \mathrm{R}dg + \mathrm{S}da + \mathrm{T}db + \mathrm{U}dc \quad \ldots\ldots \quad (15).$$

Hence we have

$$\left.\begin{array}{lll} \mathrm{P} = dw/de\,; & \mathrm{Q} = dw/df\,; & \mathrm{R} = dw/dg\,; \\ \mathrm{S} = dw/da\,; & \mathrm{T} = dw/db\,; & \mathrm{U} = dw/dc\,; \end{array}\right\} \quad (16)$$

Hence, by (14), and taking L, L to denote linear functions, we find

$$\left.\begin{array}{l} \mathrm{P} = \frac{1}{2}\mathrm{N}\Sigma\left\{\dfrac{\phi'(r)}{r}x^2 + \mathrm{L}(e, f, g, a, b, c)\right\} \\ \mathrm{S} = \frac{1}{2}\mathrm{N}\Sigma\left\{\dfrac{\phi'(r)}{r}yz + L(e, f, g, a, b, c)\right\} \end{array}\right\} \quad \ldots\ldots \quad (17),$$

and symmetrical expressions for Q, R, T, U.

§ 16. Let now our condition of zero strain be one* in which no extraneous force is required to prevent the assemblage from leaving it. We must have $\mathrm{P} = 0$, $\mathrm{Q} = 0$, $\mathrm{R} = 0$, $\mathrm{S} = 0$, $\mathrm{T} = 0$, $\mathrm{U} = 0$, when $e = 0$, $f = 0$, $g = 0$, $a = 0$, $b = 0$, $c = 0$. Hence, by (17), and the other four symmetrical formulæ, we see that

$$\left.\begin{array}{lll} \Sigma\dfrac{\phi'(r)}{r}x^2 = 0, & \Sigma\dfrac{\phi'(r)}{r}y^2 = 0, & \Sigma\dfrac{\phi'(r)}{r}z^2 = 0, \\ \Sigma\dfrac{\phi'(r)}{r}yz = 0, & \Sigma\dfrac{\phi'(r)}{r}zx = 0, & \Sigma\dfrac{\phi'(r)}{r}xy = 0 \end{array}\right\} (18).$$

Hence, in the summation for all the points x, y, z, between which and the point O there is force, we see that the first term of the summed coefficients in Q, given by (9) above, vanishes in every case, except those of fg and ea, in each of which there is only a single term; and thus from (9) and (14) we find

* The consideration of the equilibrium of the thin surface layer, in these circumstances, under the influence of merely their proper mutual forces, is exceedingly interesting, both in its relation to Laplace's theory of capillary attraction, and to the physical condition of the faces of a crystal and of surfaces of irregular fracture. But it must be deferred.

$$w = \tfrac{1}{2}\mathrm{N}\left\{\begin{array}{l}\tfrac{1}{2}e^2\Sigma\varpi\dfrac{x^4}{r^4}+(fg+\tfrac{1}{2}a^2)\,\Sigma\varpi\,\dfrac{y^2z^2}{r^4} \\ +(bc+ea)\Sigma\varpi\,\dfrac{x^2yz}{r^4}+eb\Sigma\varpi\,\dfrac{x^3z}{r^4}+\&c.\end{array}\right\} \;\ldots\; (19),$$

where $$-r\phi'(r)+r^2\phi''(r)=\varpi \;\ldots\ldots\ldots\ldots\; (20).$$

The terms given explicitly in (19) suffice to show by symmetry all the remaining terms represented by the "&c."

§ 17. Thus we see that with no limitation whatever to the number of neighbours acting with sensible force on any one point O, and with no simplifying assumption as to the law of force, we have in the quadratic for w equal values for the coefficients of fg and $\frac{1}{2}a^2$; ge and $\frac{1}{2}b^2$; ef and $\frac{1}{2}c^2$; bc and ea; ca and eb; and ab and ec. These equalities constitute the six relations promised for demonstration in § 5.

§ 18. In the particular case of an equilateral assemblage, with axes OX, OY, OZ parallel to the three pairs of opposite edges of a tetrahedron of four nearest neighbours, the coefficients which we have found for all the products except fg, ge, ef clearly vanish; because in the complete sum for a single homogeneous equilateral assemblage we have $\pm x$, $\pm y$, $\pm z$ in the symmetrical terms. Hence, and because for this case

$$\Sigma\varpi\frac{x^4}{r^4}=\Sigma\varpi\frac{y^4}{r^4}=\Sigma\varpi\frac{z^4}{r^4}, \quad\text{and}\quad \Sigma\varpi\frac{y^2z^2}{r^4}=\Sigma\varpi\frac{z^2x^2}{r^4}=\Sigma\varpi\frac{x^2y^2}{r^4} \quad (21),$$

(19) becomes

$$w=\tfrac{1}{2}\mathrm{A}\,(e^2+f^2+g^2)+\mathrm{B}\,(fg+ge+ef)+\tfrac{1}{2}n\,(a^2+b^2+c^2) \;\ldots\; (22),$$

where $$\mathrm{A}=\tfrac{1}{2}\mathrm{N}\Sigma\varpi\frac{x^4}{r^4}, \quad\text{and}\quad \mathrm{B}=n=\tfrac{1}{2}\mathrm{N}\Sigma\varpi\frac{y^2z^2}{r^4}\;\ldots\ldots\; (23).$$

§ 19. Looking to Thomson and Tait's 'Natural Philosophy,' § 695 (7),* we see that n in our present formula (22) denotes the rigidity-modulus relative to shearings parallel to the planes YOZ, ZOX, XOY; and that if we denote by n_1 the rigidity-modulus relative to shearing parallel to planes through OX, OY, OZ, and cutting (OY, OZ), (OZ, OX), (OX, OY) at angles of 45°, and if k denote the compressibility-modulus, we have

$$\left.\begin{array}{l}\mathrm{A}=k+\tfrac{4}{3}n_1; \quad \mathrm{B}=k-\tfrac{2}{3}n_1; \\ n_1=\tfrac{1}{2}(\mathrm{A}-\mathrm{B});\; k=\tfrac{1}{3}(\mathrm{A}+2\mathrm{B})\end{array}\right\}\;\ldots\ldots\ldots\; (24);$$

* This formula is given for the case of a body which is wholly isotropic in respect to elasticity moduluses; but from the investigation in §§ 681, 682 we see that our present formula, (22) or (25), expresses the elastic energy for the case of an elastic solid possessing cubic isotropy with unequal rigidities in respect to these two sets of shearings.

and our expression (22), for the elastic energy of the strained solid, becomes

$$2w = (k+\tfrac{4}{3}n_1)(e^2+f^2+g^2)+2(k-\tfrac{2}{3}n_1)(fg+ge+ef)+n(a^2+b^2+c^2) \quad \ldots\ldots (25)$$

§ 20. Using in (24) the equality $B = n$ shown in (23), we find

$$3k = 2n_1+3n \quad \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

This remarkable relation between the two rigidities and the compressibility of an equilateral homogeneous assemblage of Boscovich atoms was announced without proof in § 27 of my paper on the "Molecular Constitution of Matter."* In it n denotes what I called the facial rigidity, being rigidity relative to shearings parallel to the faces of the principal cube:† and n_1 the diagonal rigidity, being rigidity relative to shearings parallel to any of the six diagonal planes through pairs of mutually remotest parallel edges of the same cube. By (24) and (23) we see that if the law of force be such that

$$\Sigma\varpi\frac{x^4}{r^4} = 3\Sigma\varpi\frac{y^2z^2}{r^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27),$$

we have $n = n_1$, and the body constituted by the assemblage is wholly isotropic in its elastic quality. In this case (26) becomes $3k = 5n$, as found by Navier and Poisson; and thus we complete the demonstration of the statements of § 5 above.

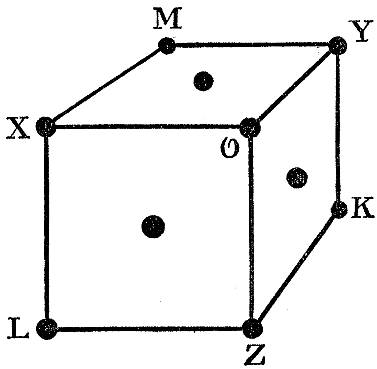
§ 21. A case which is not uninteresting in respect to Boscovichian theory, and which is very interesting indeed in respect to mechanical engineering (of which the relationship with Boscovich's theory has been pointed out and beautifully illustrated by M. Brillouin),‡ is the case of an equilateral homogeneous assemblage with forces only between each point and its twelve equidistant nearest neighbours. The annexed diagram (fig. 2) represents the point O and three of its twelve nearest neighbours (their distances λ), being in the middles of the near faces of the principal cube shown in the diagram; and three of its six next-nearest neighbours (their distances $\lambda\sqrt{2}$), being at X, Y, Z, the corners of the cube nearest to it; and, at other corners of the cube, three other neighbours K, L, M, which are next-next-next-nearest (their distances 2λ). The points in the middles of the three remote sides of the cube, not seen in the diagram, are next-next-nearest neighbours of O (their distances $\lambda\sqrt{3}$).

* 'R. S. E. Proc.,' July, 1889; Art. XCVII of my 'Math. and Phys. Papers,' vol. iii.

† That is to say, a cube whose edges are parallel to the three pairs of opposite edges of a tetrahedron of four nearest neighbours.

‡ 'Conférences Scientifiques et Allocutions' (Lord Kelvin), traduites et annotées; P. Lugol et M. Brillouin: Paris, 1893, pp. 320—325.

Fig. 2.



§ 22. Confining our attention now to O's nearest neighbours, we see that the nine not shown in the diagram are in the middles of squares obtained by producing the lines YO, ZO, XO to equal distances beyond O and completing the squares on all the pairs of lines so obtained. To see this more clearly, imagine eight equal cubes placed together, with faces in contact and each with one corner at O. The pairs of faces in contact are four squares in each of the three planes cutting one another at right angles through O; and the centres of these twelve squares are the twelve nearest neighbours of O. If we denote by λ the distance of each of them from O, we have for the coordinates x, y, z of these twelve points as follows:—

$$\left.\begin{array}{l}
\left(0,\ \frac{\lambda}{\sqrt{2}},\ \frac{\lambda}{\sqrt{2}}\right),\ \left(0,\ -\frac{\lambda}{\sqrt{2}},\ \frac{\lambda}{\sqrt{2}}\right),\ \left(0,\ \frac{\lambda}{\sqrt{2}},\ -\frac{\lambda}{\sqrt{2}}\right),\ \left(0,\ -\frac{\lambda}{\sqrt{2}},\ -\frac{\lambda}{\sqrt{2}}\right) \\
\left(\frac{\lambda}{\sqrt{2}},\ 0,\ \frac{\lambda}{\sqrt{2}}\right),\ \left(-\frac{\lambda}{\sqrt{2}},\ 0,\ \frac{\lambda}{\sqrt{2}}\right),\ \left(\frac{\lambda}{\sqrt{2}},\ 0,\ -\frac{\lambda}{\sqrt{2}}\right),\ \left(-\frac{\lambda}{\sqrt{2}},\ 0,\ -\frac{\lambda}{\sqrt{2}}\right) \\
\left(\frac{\lambda}{\sqrt{2}},\ \frac{\lambda}{\sqrt{2}},\ 0\right),\ \left(-\frac{\lambda}{\sqrt{2}},\ \frac{\lambda}{\sqrt{2}},\ 0\right),\ \left(\frac{\lambda}{\sqrt{2}},\ -\frac{\lambda}{\sqrt{2}},\ 0\right),\ \left(-\frac{\lambda}{\sqrt{2}},\ -\frac{\lambda}{\sqrt{2}},\ 0\right)
\end{array}\right\} \quad \ldots\ldots \ (28).$$

§ 23. Suppose now O to experience force only from its twelve nearest neighbours: the summations Σ of § 18 (23) will include just these twelve points with equal values of ϖ for all. These yield eight terms to $\Sigma\,(x^4/r^4)$, and four to $\Sigma\,(y^2z^2/r^4)$; and the value of each term in these sums is $\frac{1}{4}$. Thus we find that

$$A = N\varpi, \quad \text{and} \quad B = n = \tfrac{1}{2}N\varpi \ \ldots\ldots\ldots\ldots \ (29).$$

Hence and by (24), we see that

$$n_1 = \tfrac{1}{2}n \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (30).$$

Thus we have the remarkable result that, relatively to the principal cube, the diagonal rigidity is half the facial rigidity when each point experiences force only from its twelve nearest neighbours. This proposition was announced without proof in § (28) of "Molecular Constitution of Matter."*

§ 24. Suppose now the points in the middles of the faces of the cubes which in the equilateral assemblage are O's twelve equidistant nearest neighbours to be removed, and the assemblage to consist of points in simplest cubic order; that is to say, of Boscovichian points at the points of intersection of three sets of equidistant parallel planes dividing space into cubes. Fig. 2 shows O; and, at X, Y, Z, three of the six equidistant nearest neighbours which it has in the simple cubic arrangement. Keeping λ with the same signification in respect to fig. 2 as before, we have now for the coordinates of O's six nearest neighbours:

$$(\lambda\sqrt{2}, 0, 0), (0, \lambda\sqrt{2}, 0), (0, 0, \lambda\sqrt{2}),$$

$$(-\lambda\sqrt{2}, 0, 0), (0, -\lambda\sqrt{2}, 0), (0, 0, -\lambda\sqrt{2}).$$

Hence, and denoting by ϖ_1 the value of ϖ for this case, we find, by § 18 (23),

$$A = N\varpi_1 \quad \text{and} \quad B = n = 0 \quad \ldots\ldots\ldots\ldots \quad (31).$$

The explanation of $n = 0$ (facial rigidity zero) is obvious when we consider that a cube having for its edges twelve equal straight bars, with their ends jointed by threes at the eight corners, affords no resistance to change of the right angles of its faces to acute and obtuse angles.

§ 25. Replacing now the Boscovich points in the middles of the faces of the cubes, from which we supposed them temporarily annulled in § 24, and putting the results of § 23 and § 24 together, we find for our equilateral homogeneous assemblage its elasticity moduluses as follows:

$$\left.\begin{array}{l} A = N(\varpi_0 + \varpi_1) \\ B = n = \tfrac{1}{2}N\varpi_0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \quad (32),$$

where, as we see by § 16 (20) above,

$$\left.\begin{array}{l} \varpi_0 = \lambda F(\lambda) - \lambda^2 F'\lambda \\ \varpi_1 = \lambda\sqrt{2}\,F(\lambda\sqrt{2}) - 2\lambda^2 F'(\lambda\sqrt{2}) \end{array}\right\} \ldots\ldots\ldots \quad (33),$$

$F(r)$ being now taken to denote repulsion between any two of the points at any distance r, which, with $\phi(r)$ defined as in § 10, is the

* 'Math. and Phys. Papers,' vol. iii, p. 403.

meaning of $-\phi'(r)$. To render the solid, constituted of our homogeneous assemblage, elastically isotropic, we must, by § 19 (24), have $A-B = 2n$, and therefore, by (32),

$$\varpi_0 = 2\varpi_1 \dots\dots\dots\dots\dots\dots \quad (34).$$

§ 26. The last three of the six equilibrium equations § 16 (18) are fulfilled in virtue of symmetry in the case of an equilateral assemblage of single points whatever be the law of force between them, and whatever be the distance between any point and its nearest neighbours. The first three of them require in the case of § 23 that $F(\lambda) = 0$; and in the case of (24) that $F(\lambda\sqrt{2}) = 0$, results of which the interpretation is obvious and important.

§ 27. The first three of the six equilibrium equations, § 16 (18), applied to the case of § 25, yield the following equation:—

$$\sqrt{\tfrac{1}{2}}F(\lambda\sqrt{2}) = -F(\lambda) \dots\dots\dots\dots \quad (35);$$

that is to say, if there is repulsion or attraction between each point and its twelve nearest neighbours, there is attraction or repulsion of $\sqrt{2}$ of its amount between each point and its six next-nearest neighbours, unless there are also forces between more distant points. This result is easily verified by simple synthetical and geometrical considerations of the equilibrium between a point and its twelve nearest and six next-nearest neighbours in an equilateral homogeneous assemblage. The consideration of it is exceedingly interesting and important in respect to, and in illustration of, the engineering of jointed structures with redundant links or tie-struts.

§ 28. Leaving, now, the case of an equilateral homogeneous assemblage, let us consider what we may call a scalene assemblage, that is to say, an assemblage in which there are three sets of parallel rows of points, determinately fixed as follows, according to the system first taught by Bravais:—*

I. Just one set of rows of points at consecutively shortest distances λ_1.

II. Just one set of rows of points at consecutively next-shortest distances λ_2.

III. Just one set of rows of points at consecutive distances shorter than those of all other rows not in the plane of I and II.

To the condition $\lambda_3 > \lambda_2 > \lambda_1$ we may add the condition that none of the angles between the three sets of rows is a right angle, in order that our assemblage may be what we may call wholly scalene.

* 'Journal de l'École Polytechnique,' tome xix, cahier xxxiii, pp. 1—128: Paris, 1850.

§ 29. Let A′OA, B′OB, C′OC be the primary rows thus determinately found having any chosen point, O, in common; we have

$$\left.\begin{aligned} A'O = OA = \lambda_1 \\ B'O = OB = \lambda_2 \\ C'O = OC = \lambda_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (36).$$

Thus A′ and A are O's nearest neighbours; and B′ and B, O's next-nearest neighbours; and C′ and C, O's nearest neighbours not in the plane AOB. (It should be understood that there may be in the plane AOB points which, though at greater distances from O than B and B′, are nearer to O than are C and C′.)

§ 30. Supposing, now, BOC, B′OC′, &c., to be the acute angles between the three lines meeting in O; we have two equal and dichirally similar* tetrahedrons of each of which each of the four faces is a scalene acute-angled triangle. That every angle in and between the faces is acute we readily see, by remembering that OC and OC′ are shorter than the distances of O from any other of the points on the two sides of the plane AOB.†

§ 31. As a preliminary to the engineering of an incompressible elastic solid according to Boscovich, it is convenient now to consider a special case of scalene tetrahedron, in which perpendiculars from the four corners to the four opposite faces intersect in one point. I do not know if the species of tetrahedron which fulfils this condition has found a place in geometrical treatises, but I am informed by Dr. Forsyth that it has appeared in Cambridge examination papers. For my present purpose it occurred to me thus:—Let QO, QA, QB, QC be four lines of given lengths drawn from one point, Q. It is required to draw them in such relative directions that the volume of the tetrahedron OABC is a maximum. Whatever be the four given lengths, this problem clearly has one real solution and one only; and it is such that the four planes BOC, COA, AOB, ABC are cut perpendicularly by the lines AQ, BQ, CQ, OQ, respectively, each produced through Q. Thus we see that the special tetrahedron is defined by four lengths, and conclude that two equations among the six edges of the tetrahedron in general are required to make it our special tetrahedron.

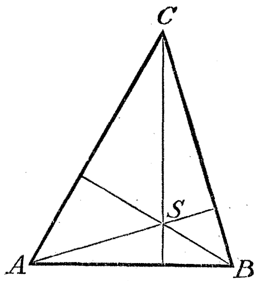
§ 32. Hence we see the following simple way of drawing a special tetrahedron. Choose as data three sides of one face and the length

* Either of these may be turned round so as to coincide with the image of the other in any plane mirror. Either may be called a pervert of the other; as, according to the usage of some writers, an object is called a *pervert* of another if one of them can be brought to coincide with the image of the other in a plane mirror (as, for example, a right hand and a left hand).

† See "Molecular Constitution of Matter," § (45), (*h*), (*i*), 'Math. and Phys. Papers,' vol. iii, pp. 412—413.

perpendicular to it from the opposite angle. The planes through this perpendicular, and the angles of the triangle, contain the perpendiculars from these angles to the opposite faces of the tetrahedron, and therefore cut the opposite sides of the triangle perpendicularly. (Thus, parenthetically, we have a proof of the known theorem of elementary geometry that the perpendiculars from the three angles of a triangle to the opposite sides intersect in one point.) Let ABC

FIG. 3.



be the chosen triangle and S the point in which it is cut by the perpendicular from O, the opposite corner of the tetrahedron. AS, BS, CS, produced through S, cut the opposite sides perpendicularly, and therefore we find the point S by drawing two of these perpendiculars and taking their point of intersection. The tetrahedron is then found by drawing through S a line SO of the given length perpendicular to the plane of ABC. (We have, again parenthetically, an interesting geometrical theorem. The perpendiculars from A, B, C to the planes of OBC, OCA, OAB cut OS in the same point; SO being of any arbitrarily chosen length.)

§ 33. I wish now to show how an incompressible homogeneous solid of wholly oblique crystalline configuration can be constructed without going beyond Boscovich for material. Consider, in any scalene assemblage, the plane of the line A′OA through any point O and its nearest neighbours, and the line B′OB through the same point and its next-nearest neighbours. To fix the ideas, and avoid circumlocutions, we shall suppose this plane to be horizontal. Consider the two parallel planes of points nearest to the plane above it and below it. The corner C of the acute-angled tetrahedron OABC, which we have been considering, is one of the points in one of the two nearest parallel planes, that above AOB we shall suppose. And the corner C′ of the equal and dichirally similar tetrahedron OA′B′C′ is one of the points in the nearest parallel plane below. All the points in the plane through C are corners of equal tetrahedrons chirally similar to OABC, and standing on the horizontal triangles oriented

as BOA. All the points C′ in the nearest plane below are corners of tetrahedrons chirally similar to OA′B′C′ placed downwards on the triangles oriented as B′OA′. The volume of the tetrahedron OABC is $\frac{1}{6}$ of the volume of the parallelepiped, of which OA, OB, OC are conterminous edges. Hence the sum of the volumes of all the upward tetrahedrons having their bases in one plane is $\frac{1}{6}$ of the volume of the space between large areas of these planes: and, therefore, the sum of all the chirally similar tetrahedrons, such as OABC, is $\frac{1}{6}$ of the whole volume of the assemblage through any larger space. Hence any homogeneous strain of the assemblage which does not alter the volume of the tetrahedrons does not alter the volume of the solid. Let tie-struts OQ, AQ, BQ, CQ be placed between any point Q within the tetrahedron and its four corners, and let these tie-struts be mechanically jointed together at Q, so that they may either push or pull at this point. This is merely a mechanical way of stating the Boscovichian idea of a second homogeneous assemblage, equal and similarly oriented to the first assemblage and placed with one of its points at Q, and the others in the other corresponding positions relatively to the primary assemblage. When it is done for all the tetrahedrons chirally similar to OABC, we find four tie-strut ends at every point O, or A, or B, or C, for example, of the primary assemblage. Let each set of these four ends be mechanically jointed together, so as to allow either push or pull. A model of the curious structure thus formed was shown at the conversazione of the Royal Society of June 7, 1893. It is for three dimensions of space what ordinary hexagonal netting is in a plane.

§ 34. Having thus constructed our model, alter its shape until we find its volume a maximum. This brings the tetrahedron, OABC, to be of the special kind defined in § 30. Suppose for the present the tie-struts to be absolutely resistant against push and pull, that is to say, to be each of constant length. This secures that the volume of the whole assemblage is unaltered by any infinitesimal change of shape possible to it; so that we have, in fact, the skeleton of an incompressible and inextensible solid.* Let now any forces whatever, subject to the law of uniformity in the assemblage, act between the points of our primary assemblage: and, if we please, also between all the points of our second assemblage; and between all the points of the two assemblages. Let these forces fulfil the conditions of equilibrium; of which the principle is described in § 16 and applied to find the equations of equilibrium for the simpler case of a single homogeneous assemblage there considered. Thus we have an incompressible elastic

* This result was given for an equilateral tetrahedronal assemblage in § 67 of "Molecular Constitution of Matter," 'Math. and Phys. Papers,' vol. iii, pp. 425—426.

solid; and, as in § 17 above, we see that there are fifteen independent coefficients in the quadratic function of the strain-components expressing the work required to produce an infinitesimal strain. Thus we realise the result described in § 7 above.

§ 35. Suppose now each of the four tie-struts to be not infinitely resistant against change of length, and to have a given modulus of longitudinal rigidity, which, for brevity, we shall call its stiffness. By assigning proper values to these four stiffnesses, and by supposing the tetrahedron to be freed from the two conditions making it our special tetrahedron, we have six quantities arbitrarily assignable, by which, adding these six to the former fifteen, we may give arbitrary values to each of the twenty-one coefficients in the quadratic function of the six strain-components with which we have to deal when change of bulk is allowed. Thus, in strictest Boscovichian doctrine, we provide for twenty-one independent coefficients in Green's energy-function. The dynamical details of the consideration of the equilibrium of two homogeneous assemblages with mutual attraction between them, and of the extension of §§ 9—17 to the larger problem now before us, are full of purely scientific and engineering interest, but must be reserved for what I hope is a future communication.

II. "Magnetic Qualities of Iron." By J. A. Ewing, M.A., F.R.S., Professor of Mechanism and Applied Mechanics in the University of Cambridge, and Miss Helen G. Klaassen, Lecturer in Physics, Newnham College. Received June 7, 1893.

(Abstract.)

The paper describes a series of observations of magnetic quality in various specimens of sheet iron and iron wire. A principal object was to determine the amount of energy lost in consequence of magnetic hysteresis when the iron under examination was carried through cyclic magnetising processes between assigned limits of the magnetic induction B. For this purpose observations of the relation of the induction B to the magnetic force H were made, from which curves were drawn, and the area enclosed by the curves in cyclic magnetising processes was measured. Many such cycles were gone through in the case of each of the specimens, the limits between which B was reversed being varied step by step in successive cycles, to allow the relation of the energy expended or of $\int H dI$ to B to be determined. The curves of B and H in these graded cycles are drawn in the paper, as well as curves showing the relation of $\int H dI$ to B and to H. Most of these experiments were made by the ballistic method, the specimens