Date.	Declination.		Horizontal force.		Dip.	
	L.M.T.	Obs. result.	L.M.T.	Obs. result.	L.M.T.	Obs. result.
1893.				c.g.s.		
April 4	9.1 A.M.	18° 45′ W.	5.13 P.M.	0·30400		
„ 5	8.31 „	18° 42′ „	9.26 A.M.	0·30434		
„ 14	8.39 „	18° 45′ „	8.53 „	0·30394		
„ 14	..	..	..	..	9.6 A.M.	Needle 1, 29° 9′·1
„ 14	..	..	..	..	9.28 „	„ 2, 29° 8′·2
Means ..	..	18° 44′ „	..	0·30409	..	29° 8′·7

Bathurst, *River Gambia*, lat. 13° 28′ N., long. 16° 37′ W.

The station was on a large piece of open ground and near the centre of McCarthy Square. All the observations taken were made on April 20, 1893.

Declination....	at 8.16 A.M.	L.M.T.	= 18° 50′ W.
Horizontal force	at 8.44	„	= 0·30514 c.g.s.
Dip	at 8.17	„	= Needle 1, 28° 43′·4.
„	at 9.14	„	= „ 2, 28° 42′·1.

III. "A certain Class of Generating Functions in the Theory of Numbers." By Major P. A. MacMahon, R.A., F.R.S. Received November 3, 1893.

(Abstract.)

The present investigation arose from my "Memoir on the Compositions of Numbers," recently read before the Royal Society and now in course of publication in the 'Philosophical Transactions.' The main theorem may be stated as follows:—

If $X_1, X_2, \ldots, X_n$ be linear functions of quantities $x_1, x_2, \ldots, x_n$ given by the matricular relation

$$(X_1, X_2, \ldots, X_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & {}_{n3} & a_{nn} \end{pmatrix} (x_1, x_2, \ldots, x_n),$$

that portion of the algebraic fraction

$$\frac{1}{(1-s_1X_1)\,(1-s_2X_2)\,\ldots\,(1-s_nX_n)},$$

which is a function of the products,

$$s_1x_1,\ s_2x_2,\ \ldots,\ s_nx_n,$$

only, is $1/V_n$, where (putting $s_1 = s_2 = \ldots = s_n = 1$)

$$V_n = (-)^n x_1x_2\ldots x_n \begin{vmatrix} a_{11}-1/x_1, & a_{12}, & \cdots & a_{1n} \\ a_{21}, & a_{22}-1/x_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-1/x_n \end{vmatrix}$$

The proof of this theorem rests upon an identity which, for order 3, is

$$\begin{vmatrix} a_{11}s_1x_1-1, & a_{12}s_1x_1, & a_{13}s_1x_1, \\ a_{21}s_2x_2, & a_{22}s_2x_2-1, & a_{23}s_2x_2, \\ a_{31}s_3x_3, & a_{32}s_3x_3, & a_{33}s_3x_3-1, \end{vmatrix}$$

$$= \begin{vmatrix} 1-s_1X_1, & 0, & 0, \\ 0, & 1-s_2X_2, & 0, \\ 0, & 0, & 1-s_3X_3, \end{vmatrix}$$

$$\times \begin{vmatrix} \frac{s_1(a_{11}x_1-X_1)}{1-s_1X_1}-1, & \frac{a_{12}s_1x_1}{1-s_1X_1}, & \frac{a_{13}s_1x_1}{1-s_1X_1}, \\ \frac{a_{21}s_2x_2}{1-s_2X_2}, & \frac{s_2(a_{22}x_2-X)}{1-s_2X_2}-1, & \frac{a_{23}s_2x_2}{1-s_2X_2}, \\ \frac{a_{31}s_3x_3}{1-s_3X_3}, & \frac{a_{32}s_3x_3}{1-s_3X_3}, & \frac{s_3(a_{33}x_3-X_3)}{1-s_3X_3}-1, \end{vmatrix}$$

and is very easily established.

An instantaneous deduction of the general theorem is the result that the generating function for the coefficients of $x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$ in the product

$$X_1^{\xi_1}X_2^{\xi_2}\ldots X_n^{\xi_n}$$

is

$$1/V_n.$$

The expression V_n involves the several coaxial minors of the determinant of the linear functions. Thus

$$V_3 = 1-a_{11}x_1-a_{22}x_2-a_{33}x_3+|a_{11}a_{22}|x_1x_2+|a_{11}a_{33}|x_1x_3+|a_{22}a_{33}|x_2x_3$$
$$-|a_{11}a_{22}a_{33}|x_1x_2x_3.$$

The theorem is of considerable arithmetical importance and is also of interest in the algebraical theories of determinants and matrices.

The product

$$X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n}$$

often appears in arithmetic as a redundant form of generating function. The theorem above supplies a condensed or exact form of generating function.

Ex. gr. It is clear that the number of permutations of the $\Sigma\xi$ symbols in the product

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$$

which are such that every symbol is displaced is obviously the coefficient of

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$$

in the product

$$(x_2 + \dots + x_n)^{\xi_1} (x_1 + x_3 + \dots + x_n)^{\xi_2} \dots (x_1 + x_2 + \dots + x_{n-1})^{\xi_n},$$

and thence we easily pass to the true generating function

$$\frac{1}{1 - \Sigma x_1 x_2 - 2\Sigma x_1 x_2 x_3 - 3\Sigma x_1 x_2 x_3 x_4 - \dots - (n-1)\, x_1 x_2 \dots x_n}.$$

In the paper many examples are given.

Frequently the redundant and condensed generating functions are differently interpretable; we then obtain an arithmetical correspondence, two cases of which presented themselves in the "Memoir on the Compositions of Numbers."

A more important method of obtaining arithmetical correspondences is developed in the researches which follow the statement and proof of the theorem.

The general form of V_n is such that the equation

$$V_n = 0$$

gives each quantity x_s as a homographic function of the remaining $n-1$ quantities, and it is interesting to enquire whether, assuming the coefficients of V_n arbitrarily, it is possible to pass to a corresponding redundant generating function.

I find that the coefficients of V_n must satisfy

$$2^n - n^2 + n - 2$$

conditions, and, assuming the satisfaction of these conditions, a redundant form can be constructed which involves

$$n-1$$

undetermined quantities. In fact, when a redundant form exists at all, it is necessarily of a $(n-1)$tuply infinite character.

We are now able to pass from any particular redundant generating function to an equivalent generating function which involves $n-1$ undetermined quantities. Assuming these quantities at pleasure, we obtain a number of different algebraic products, each of which may have its own meaning in arithmetic, and thus the number of arithmetical correspondences obtainable is subject to no finite limit.

This portion of the theory is given at length in the paper, with illustrative examples.

Incidentally interesting results are obtained in the fields of special and general determinant theory. The special determinant, which presents itself for examination, provisionally termed "inversely symmetric," is such that the constituents symmetrically placed in respect to the principal axis have, each pair, a product unity, whilst the constituents on the principal axis itself are all of them equal to unity. The determinant possesses many elegant properties which are of importance to the principal investigation of the paper. The theorems concerning the general determinant are connected entirely with the co-axial minors.

I find that the general determinant of even order, greater than two, is expressible in precisely two ways as an irrational function of its co-axial minors, whilst no determinant of uneven order is so expressible at all.

Of order superior to 3, it is not possible to assume arbitrary values for the determinant itself and all of its co-axial minors. In fact of order n the values assumed must satisfy

$$2^n - n^2 + n - 2$$

conditions, but, these conditions being satisfied, the determinant can be constructed so as to involve $n-1$ undetermined quantities.

IV. "On the Whirling and Vibration of Shafts." By STANLEY DUNKERLEY, M.Sc., Berkeley Fellow of the Owens College, Manchester. Communicated by OSBORNE REYNOLDS, F.R.S. Received July 13, 1893.

(Abstract.)

It is well known that every shaft, however nearly balanced, when driven at a particular speed bends, and, unless the amount of deflection be limited, might even break, although at higher speeds the shaft again runs true. The particular or "critical" speed depends on the manner in which the shaft is supported, its size and modulus of elasticity, and the size, weights and positions, of any pulleys it carries.