This is below the experimental value for round pipes, and is about half what might be expected to be the experimental value for a flat pipe, which leaves a margin to meet the other kinematical conditions for steady mean-mean-motion.

(*o*.) That the discriminating equation also affords a definite expression for the resistance, which proves that, with smooth fixed boundaries, the conditions of dynamical similarity under any geometrical similar circumstances depend only on the value of

$$\frac{\rho}{\mu^2}\frac{dp}{dx}b^3,$$

where b is one of the lateral dimensions of the pipe; and that the expression for this resistance is complex, but shows that above the critical velocity the relative-mean-motion is limited, and that the resistances increase as a power of the velocity higher than the first.

III. "On certain Functions connected with Tesseral Harmonics, with Applications." By A. H. Leahy, M.A., late Fellow of Pembroke College, Cambridge, Professor of Mathematics at Firth College, Sheffield. Communicated by Professor W. M. Hicks, F.R.S. Received March 24, 1894.

(Abstract.)

The transformation of a zonal harmonic referred to a pole on a sphere to another pole on the same sphere, and its expression in a series containing the $2n+1$ harmonics of the same order referred to this new pole, is an operation frequently employed in physical research. The purpose of this paper is the investigation of certain functions of the angular distance between the poles which occur when a general tesseral harmonic is transformed from one pole and plane to another pole and another plane of reference. If the coordinates of any point on the sphere when referred to the first pole are β' and γ'; β' denoting the colatitude, and γ' the longitude; and if the coordinates of the same point when referred to the second pole are δ' and q; δ' denoting the colatitude and q the longitude referred to a plane through the two poles, it is shown that

$$\mathrm{P}(n,m,\mu')\cos m\gamma' = \cos m\gamma\left\{u_{m,o}\,.\,\mathrm{P}_n(\nu) + 2\Sigma\frac{n-r\,!}{n+r\,!}u_{mr}\,.\,\mathrm{P}(n,r,\nu)\cos rq\right\}$$

$$+\sin m\gamma\,.\,2\Sigma\frac{n-r\,!}{n+r\,!}v_{mr}\,.\,\mathrm{P}(n,r,\nu)\sin rq,$$

$$P(n,m,\mu')\sin m\gamma' = \sin m\gamma\left\{u_{m,o}\,.\,P_n(\nu) + 2\Sigma\frac{n-r\,!}{n+r\,!}\,u_{mr}\,.\,P(n,r,\nu)\cos rq\right\}$$

$$-\cos m\gamma\,.\,2\Sigma\frac{n-r\,!}{n+r\,!}\,v_{mr}\,.\,P(n,r,\nu)\sin rq,$$

where $P(n,m,\mu')$ is the "associated function" $\dfrac{d^mP_n(\mu')}{d\mu'^m}\,.\,(1-\mu'^2)^{m/2}$; μ' and ν are put for $\cos\beta'$, $\cos\delta'$, γ is the longitude of the second pole referred to the original pole and plane, and u_{mr} v_{mr} are the functions of β, the angular distance between the poles whose properties are discussed in the paper. When m is zero, *i.e.*, when $P(n,m,\mu')$ is a zonal harmonic, the function u_{mr} reduces to $P(n,r,\mu)$, if μ is put for $\cos\beta$, and v_{mr} is zero. The general equations connecting the functions, and the values of the functions for general and for particular values of m and r are investigated.

If δ' and ϵ' are the colatitude and longitude of a point referred to the second pole and any plane through the pole, the integral of the product of any two tesseral harmonics both of the nth order over the surface of a sphere can be expressed concisely in terms of the functions u_{mr}, v_{mr}. The result is

$$\iint P(n,m,\mu')\cos(m\gamma'+\alpha)\,.\,P(n,m,\nu)\cos(r\epsilon'+\rho)\,dS =$$

$$\frac{4\pi a^2}{2n+1}\{\cos(m\gamma+\alpha)\cos(r\epsilon+\rho)\,u_{mr}(\beta)-\sin(m\gamma+\alpha)\sin(r\epsilon+\rho)v_{mr}(\beta)\}$$

if γ is the longitude of the second pole referred to the plane through the first, ϵ the longitude of the first pole referred to the plane through the second, β is the angular distance between the poles, and α, ρ are constants.

The functions u_{mr}, v_{mr} are connected by several equations, bearing a great resemblance to equations connecting tesseral harmonics of the same order. They are of course functions of n, and should be written, when n may have different values, in the form $u_{n,m,r}, v_{n,m,r}$, but the n is omitted for brevity in most of the results. Some of the most important results are the following, the dashes denoting differential coefficients—

$$u''_{mr}\sin\beta + u'_{mr}\cos\beta + \{n(n+1)\sin\beta - (m^2+r^2)\operatorname{cosec}\beta\}u_{mr}$$

$$= 2mrv_{mr}\cot\beta \;\ldots\ldots\; (21);$$

$$u_{m,r+1} + 2u'_{mr} - (n+r)(n-r+1)\,u_{m,r-1} = 0 \;\ldots\ldots\; (24);$$

$$u_{m,r+1} - 2r\cot\beta\,u_{mr} + (n+r)(n-r+1)\,u_{m,r-1} = 2m\operatorname{cosec}\beta\,v_{mr}$$

$$\ldots\ldots\; (27).$$

All the relations connecting u_{mr}, v_{mr}, &c., are duplicate ones, similar relations being obtained by interchanging u and v.

The differential equation satisfied by either function is of the fourth order, the two functions being different solutions of this equation. The two remaining solutions of the equation have also been obtained, and called "functions of the second kind." The equation of finite differences satisfied by the functions is also of the fourth order.

The general value of u_{mr} is—

$$2u_{mr} = \mathrm{P}(n, m+r, \mu)$$

$$+ \sum_{k=1}^{k=\mathrm{I}(r/2)} (-1)^k . rm\,(m+r-2k)\,\mathrm{P}(n, m+r-2k, \mu)$$

$$\times \sum_{s=1}^{s=k} (-1)^s \frac{m-k+s-1\,!\; m+r-k-1\,!\; n+k-s\,!\; r-k+s-1\,!\; k-1\,!}{m-k\,!\; m+r-k-s\,!\; n-k+s\,!\; r-k\,!\; s-1\,!\; k-s\,!\; s\,!}$$

$$+ \sum_{k=\mathrm{I}(r/2)+1}^{k=r-1} (-1)^k rm\,(m+r-2k)\,\frac{n+m\,!\; n-m+2k-r\,!}{n-m\,!\; n+m-2k+r\,!}\,\mathrm{P}(n, m+r-2k, \mu)$$

$$\times \sum_{s=1}^{s=r-k} (-1)^s \frac{m-k+s-1\,!\, m+r-k-1\,!\, n+r-k-s\,!\, r-k+1\,!\, k+s-1\,!}{m-k\,!\, m+r-k-s\,!\, n-r+k+s\,!\, r-k+s\,!\, s\,!\, k\,!\, s-1\,!}$$

$$+ (-1)^r . \frac{n+m\,!\; n-m+r\,!}{n-m\,!\; n+m-r\,!}\,\mathrm{P}(n, m-r, \mu),$$

where $\mathrm{I}(r/2)$ is the greatest integer in $r/2$.

The value of v_{mr} is given by

$$v_{mr} . \sin\beta = \sum_{k=0}^{k=\mathrm{I}\left(\frac{r-1}{2}\right)} (-1)^k (m+r-2k-1)\,\mathrm{P}(n, m+r-2k-1, \mu)$$

$$\times \sum_{s=0}^{s=k} (-1)^s \frac{m-k+s-1\,!\; m+r-k-1\,!\; n+k-s\,!\; r-k+s-1\,!\; k\,!}{m-k-1\,!\; m+r-k-s-1\,!\; n-k+s\,!\; r-k-1\,!\; s\,!\; k-s\,!\; s\,!}$$

$$+ \sum_{k=\mathrm{I}\left(\frac{r-1}{2}\right)+1}^{k=r-1} (-1)^k (m+r-2k-1)\,\frac{n+m\,!\; n-m+2k-r-1\,!}{n-m\,!\; n+m-2k+r+1\,!}\,\mathrm{P}(n, m+r-2k-1, \mu)$$

$$\sum_{s=0}^{s=r-k-1} (-1)^s \frac{m-k+s-1\,!\; m+r-k-1\,!\; n+r-k-s-1\,!\; r-k-1\,!\; k+s\,!}{m-k-1\,!\; m+r-k-s-1\,!\; n-r+k+s+1\,!\; r-k-s-1\,!\; s\,!\; k\,!\; s\,!}$$

Simpler values for u_{mr} v_{mr} are given for general values of m from $r = 0$ to $r = 6$ inclusive.

The values of the functions u_{mr}, v_{mr} are of a simpler form when β is

a right angle, and can be expressed by a single series. When $n-r$ is even, the series

$$(-1)^{\frac{n-r}{2}} \frac{\frac{n-r}{2}\,!\,\frac{n+r}{2}\,!\,m\,!\,m\,!}{n-r\,!\,2m\,!}\,2^{-n}$$

$$\sum_{t=0} (-1)^t \frac{n+2m-r-2t\,!\,n+r-2t\,!}{m-t\,!\,t\,!\,\frac{n+r}{2}+t\,!\,\frac{n-r}{2}-t\,!\,\frac{n+r}{2}-m+t\,!\,\frac{n-r}{2}+m-t\,!}$$

is the value of $u_{mr}(\pi/2)$ when m is even, and of $v_{mr}(\pi/2)$ when m is odd; the series being continued until one of the factorials in the denominator becomes negative; and n being supposed greater than $2m$. When n is less than $2m$, the lower limit of t is $m-\frac{1}{2}(n+r)$.

A similar series gives the values of $u_{mr}(\pi/2)$ when m is odd, and of $v_{mr}(\pi/2)$ when m is even for the case when $n-r$ is odd. The values of $u_{mr}(\pi/2)$ when m is even and of $v_{mr}(\pi/2)$ when m is odd, are in this case equal to zero.

When $n-r$ is even, the values of $u_{mr}(\pi/2)$ when m is odd, and of $v_{mr}(\pi/2)$ when m is even, are also equal to zero.

The value of u_{mr} is in all cases equal to u_{rm}, and the value of v_{mr} is equal to v_{rm}. This result gives several algebraic identities, using general values of u_{mr}. Since $u_{o,r} = \mathrm{P}(n,r,\mu)$, we have by this result $u_{m,o} = \mathrm{P}(n,m,\mu)$, whence we get the result that

$$\int \mathrm{P}\,(n,m,\mu') \cos m\gamma' dq = 2\pi \mathrm{P}_n\,(\nu)\,.\,\mathrm{P}\,(n,m,\mu) \cos m\gamma.$$

Thus the line integral of a Laplace's function referred to the first pole along a small circle described about the second pole at angular distance δ from it is the value of the function at the second pole multiplied by $2\pi\mathrm{P}_n(\nu)\,.\sin\delta$, where ν is $\cos\delta$.

Equations can also be obtained connecting $u_{n,m,r}$ and $u_{n+1m,r}$, where the n's are different. The most useful result is

$$\begin{aligned} n\,(n-m+1)\,(n-r+1)\,u_{n+1,m,r} &- (2n+1)\,n\,(n+1)\cos\beta u_{n,m,r} \\ &+ (n+1)\,(n+m)\,(n+r)\,u_{n-1,m,r} = (2n+1)\,mrv_{n,m,r} \quad \ldots\ldots \quad (45), \end{aligned}$$

and a similar equation obtained by interchanging u and v. From this equation a table of the functions for different values of n can be calculated, and is given from $n=0$ to $n=4$. Since $v_{m,o} = v_{o,m} = 0$, the number of the functions for any given value of n is $(n+1)^2+n^2$.

Two physical applications of the results are given. The first is an application of the result of a line integral of a Laplace's function referred to one pole along a small circle described about another. The result is employed to establish that the law assumed by Boltzmann and Maxwell for the number of particles which have a given velocity in

an irregular system of moving molecules (or a "disturbed gas") is unaltered in form by collisions between the molecules. In the second application the functions are used to find the mutual potential energy of two layers of gravitating matter on two spheres, the density at any point on each sphere being expressed in terms of spherical harmonics referred to fixed coordinates upon it, and the spheres having any position with reference to the line joining their centres. The case of two ellipsoids not differing much from spheres is also worked out numerically, and the stable positions discussed. A stable orbit is possible with the major axes of the ellipsoids constantly in a straight line. If one ellipsoid is fixed and the other projected so as to describe a nearly circular orbit about it, with its major axis initially pointing to the centre of the other, the orbit will be possible if in a plane perpendicular to the least axis of the greater, but the deviation of the major axis of the second from the line of centres will contain a term which to the first approximation is secular, and may ultimately cause this axis to deviate from its initial position. There are three stable positions for the second ellipsoid if the first ellipsoid is fixed and the centre of the other fixed. These positions will in general be with the major axis of the second pointing towards the centre of the first, and in a line with the major, mean, and least axes of the first, but if c, the distance between the centres, is so small that

$$5\left(\frac{2}{a_1^2}-\frac{1}{a_1'^2}-\frac{1}{a_1''^2}\right)c^2 < \left(\frac{12}{a_1^2}-\frac{7}{a_1''^2}-\frac{5}{a_1'^2}\right)a_1^2, \text{ or than } \left(\frac{12}{a_1'^2}-\frac{7}{a_1''^2}-\frac{5}{a_1^2}\right)a_1^2,$$

where $a_1a_1'a_1''$ are the least, mean, and greatest axes of the first sphere, the stable positions will be different. Thus the stable positions will always be with major axis of the second in the line of centres if c^2/a_1^2 is greater than 7/5.

The "functions of the second kind," which are the two remaining solutions of the differential equation of the fourth order satisfied by u_{mr} v_{mr}, are also briefly investigated.

IV. "On the Measurement of the Magnetic Properties of Iron." By THOMAS GRAY, B.Sc., F.R.S.E., Professor of Dynamic Engineering, The Rose Polytechnic Institute, Terre Haute, Indiana. Communicated by Lord KELVIN, P.R.S. Received April 6, 1894.

(Abstract.)

This paper gives the results of a continuation of the investigation which formed the subject of a paper communicated to the Royal Society in 1892, and published in the 'Philosophical Transactions,'