Hence, the effect of elastic distortion is to prolong the period in the ratio $1+\epsilon'/\epsilon : 1$. Taking a spheroid of the same size and mean density as the Earth rotating in a sidereal day, we find that the period is extended from 232 days to 335 days if the rigidity be that of steel. Assuming that in the case of the actual Earth the effects of heterogeneity are to still further prolong the period in the ratio 305 : 232, we find for the period of oscillation of the Earth, supposed of the rigidity of steel, 440 days.

Now, observation* indicates that the earth is undergoing such an oscillation as we have been discussing in a period of 427 days instead of the period of 305 days indicated by theory on the assumption that the motion would take place sensibly in the same manner as if the earth were rigid. We see now that the divergence between theory and observation can be explained by taking into account the elastic distortions of the solid parts of the earth, and that the degree of rigidity required to account for the observed period is slightly in excess of the rigidity of steel.

This explanation has been previously offered by Professor Newcomb,† who has treated the problem by a simple geometrical method. The main object in undertaking the present analytical investigation was to examine the validity of certain hypotheses made by Newcomb. An examination of the type of oscillation indicated by our analysis shows that the procedure adopted by Newcomb is legitimate, but that there is a slight error in his assumed law of displacement of the pole of figure due to centrifugal force. Our method has the additional advantage of showing the degree of accuracy to which the results may be expected to hold good.

## II. "On a Type of Spherical Harmonics of unrestricted Degree, Order, and Argument." By E. W. Hobson, Sc.D., F.R.S. Received December 23, 1895.

(Abstract.)

The type of harmonics considered in this memoir is

$$r^n \begin{matrix}\cos \\ \sin\end{matrix} m\phi \,.\, u_n{}^m(\mu),$$

where $u_n{}^m(\mu)$ satisfies the differential equation

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}u = 0,$$

* *Vide* Chandler, 'Astron. Journal,' vols. 11, *et seq.*

† 'Monthly Notices of the Royal Astron. Soc.,' March, 1892.

known as the differential equation of Legendre's associated functions; the degree $n$, the order $m$, and the argument $\mu$ are not, as in the case of the ordinary system of spherical harmonics, restricted to be real and such that $n$ and $m$ are integral and $\mu$ is a proper fraction, but are supposed to have unrestricted real or complex values. The investigation is undertaken with the object of bringing the various types of harmonics, such as toroidal functions, conal harmonics, &c., under one general treatment.

The two particular functions $P_n^m(\mu)$, $Q_n^m(\mu)$, which satisfy the above differential equation are first defined in such a manner that they are uniform over the whole $\mu$-plane, which, however, has a cross-cut extending along the real axis from $\mu = 1$ to $\mu = -\infty$.

The definitions obtained are the following—

$$P_n^m(\mu) = \frac{e^{-n\pi\iota}}{4\pi \sin n\pi} \cdot \frac{\Pi(n+m)}{2^n \Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int^{(\mu+,\, 1+,\, \mu-,\, 1-)} (t^2-1)^n (t-\mu)^{-n-m-1} dt,$$

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \cdot \frac{\Pi(n+m)}{2^n \Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int^{(-1+,\, 1-)} (t^2-1)^n (t-\mu)^{-n-m-1} dt,$$

where in $(\mu^2-1)^{\frac{1}{2}m}$, the phases of $\mu-1$, $\mu+1$, are both zero when $\mu$ is real and greater than unity, and each varies between the values $\pm\pi$ for various positions of the point $\mu$. Precise definitions are given of the meanings to be attached to the integrands. The path of integration in the case of $P_n^m(\mu)$ consists of a loop described in the positive direction round the point $t = \mu$, followed by one in the positive direction round the point $t = 1$, then a loop in the negative direction round the point $t = \mu$, and finally a loop in the negative direction round the point $t = 1$, the whole forming a closed path, *i.e.*, one for which the integrand attains its initial value after a complete description. In the case of $Q_n^m(\mu)$ only two loops are required to form the closed path, one described positively round $t = -1$, followed by one described negatively round $t = +1$. These definitions are so chosen that in the case of real integral values of $n$ and $m$, the functions coincide with the ordinary well-known Legendre's associated functions.

From these definitions the following representations of the functions by series are deduced—

$$P_n^m(\mu) = \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right) \left(\text{mod. } \frac{1-\mu}{2} < 1\right),$$

$$Q_n^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi} \cdot \frac{1}{\Pi(-m)} \left\{ e^{\mp n\pi\iota} \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1-\mu}{2}\right) - \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right) \right\},$$

where mod. $\left(\frac{1 \pm \mu}{2}\right) < 1$; the upper or lower sign is to be taken in the exponential, according as the imaginary part of $\mu$ is positive or negative. Degenerate forms of these series for special restrictions as to $n$ and $m$ are considered.

The following expressions for $P_n^m(\mu)$, $Q_n^m(\mu)$ when mod. $\mu > 1$ are obtained—

$$P_n^m(\mu) = \frac{\sin(n+m)\pi}{2^{n+1}\cos n\pi} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \mu^{-n-m-1}$$

$$F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{\mu^2}\right) + 2^n \frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \mu^{n-m}$$

$$F\left(\frac{m-n+1}{2}, \frac{m-n}{2}, \frac{1}{2}-n, \frac{1}{\mu^2}\right),$$

$$Q_n^m(\mu) = \frac{e^{m\pi\iota}}{2^{n+1}} \frac{\Pi(n+m)\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \mu^{-n-m-1}$$

$$F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{\mu^2}\right).$$

The following relations between the particular integrals of the differential equation obtained by changing $n$ into $-n-1$, and $m$ into $-m$, are found; by means of these relations the eight solutions are expressed in terms of two of them—

$$P_n^m(\mu) = P_{-n-1}^m(\mu), \quad \frac{e^{-m\pi\iota} Q_n^m(\mu)}{\Pi(n+m)} = \frac{e^{m\pi\iota} Q_n^{-m}(\mu)}{\Pi(n-m)},$$

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi \cos n\pi} \{Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi\},$$

$$P_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)}\{P_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, Q_n^m(\mu)\}.$$

It is further shown that

$$P_n^m(-\mu) = e^{\mp n\pi\iota} P_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} \,.\, e^{-m\pi\iota} Q_n^m(\mu),$$

$$Q_n^m(-\mu) = -e^{\pm n\pi\iota} Q_n^m(\mu),$$

where the upper or lower sign in the exponentials is taken according as the imaginary part of $\mu$ is positive or negative.

The following expressions for the functions are obtained for the domain of $\mu = 0$:—

$$\mathrm{P}_n^m(\mu) = e^{\mp m\pi\iota}2^m \cos\frac{n+m}{2}\pi \frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}$$
$$\mathrm{F}\left(\frac{m+n+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$
$$+\, e^{\mp m\pi\iota}2^m \sin\frac{n+m}{2}\pi \frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu$$
$$\mathrm{F}\left(\frac{m+n+2}{2}, \frac{m-n+1}{2}, \tfrac{3}{2}, \mu^2\right);$$

the upper or lower sign is to be taken in the exponential, according as $\mu$ is above or below the real axis;

$$\mathrm{Q}_n^m(\mu) = \frac{\iota a}{2}\,.\,2^m \frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\,\mathrm{F}\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$
$$+\, b\,.\,2^m \frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\mu\;\mathrm{F}\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right),$$

where $a = -e^{-(m-n)\frac{1}{2}\pi\iota}$, $b = e^{(m-n)\frac{1}{2}\pi\iota}$, if $\mu$ is above the real axis; and $a = e^{(3m+n)\frac{1}{2}\pi\iota}$, $b = e^{(3m+n)\frac{1}{2}\pi\iota}$, if $\mu$ is below the real axis.

Special conventions are made as to the values to be attached to the functions at points in the cross-cut between the points $\pm 1$; these conventions are so made that the values of the functions shall be real for real values of $m$ and $n$; they are given by

$$\mathrm{P}_n^m(\cos\theta) = e^{\frac{1}{2}m\pi\iota}\,\mathrm{P}_n^m(\cos\theta+0\,.\,\iota) = e^{-\frac{1}{2}m\pi\iota}\,\mathrm{P}_n^m(\cos\theta-0\,.\,\iota),$$

$$e^{m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta) = \tfrac{1}{2}\{e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta+0\,.\,\iota)+e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta-0\,.\,\iota)\};$$

it is further shown that

$$e^{-\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta+0\,.\,\iota)-e^{\frac{1}{2}m\pi\iota}\,\mathrm{Q}_n^m(\cos\theta-0\,.\,\iota) = -\iota\pi e^{m\pi\iota}\,\mathrm{P}_n^m(\cos\theta).$$

The following expressions are obtained for the functions in series of powers of $\mu-\sqrt{\mu^2-1}$:—

$$P_n^m(\mu) = 2^m \frac{\sin(n+m)\pi}{\cos n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} z^{-n-m-1}$$

$$F\left(\tfrac{1}{2}+m, n+m+1, n+\tfrac{3}{2}, \frac{1}{z^2}\right)$$

$$+2^m \frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} z^{n-m} F\left(\tfrac{1}{2}+m, m-n, \tfrac{1}{2}-n, \frac{1}{z^2}\right),$$

$$Q_n^m(\mu) = 2^m e^{m\pi\iota} \frac{\Pi(n+m)\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} z^{-(n+m+1)}$$

$$F\left(\tfrac{1}{2}+m, n+m+1, n+\tfrac{3}{2}, \frac{1}{z^2}\right),$$

where $z$ denotes $\mu+\sqrt{\mu^2-1}$.

A second expression for each of the functions is obtained in the form of an integral, which might serve as an alternative definition of the functions—

$$P_n^m(\mu) = \frac{1}{2\pi\iota} \cdot 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int^{(z+,\, z^{-1}-)} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh,$$

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota} \cdot 2^m \cdot \frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi} (\mu^2-1)^{\frac{1}{2}m}$$

$$\int^{(z^{-1}+,\, o+,\, z^{-1}-,\, o-)} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh,$$

the meanings of the integrands being as before precisely defined; these expressions are not readily deducible from the former ones.

Expansions in powers of $\dfrac{\mu \pm \sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$, are obtained for the two functions, and the following special cases are deduced :—

$$P_n^m(\cos\theta) = \frac{2}{\sqrt{\pi}} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} \left[ \frac{\cos\left(\overline{n+\frac{1}{2}}\,\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}} + \frac{1^2-4m^2}{2.2n+3} \right.$$

$$\left. \frac{\cos\left(\overline{n+\frac{3}{2}}\,\theta - \frac{3\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}} + \frac{1^2-4m^2.3^2-4m^2}{2.4.2n+3.2n+5} \frac{\cos\left(\overline{n+\frac{5}{2}}\,\theta - \frac{5\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{5}{2}}} + \cdots \right]$$

This series represents the function for unrestricted values of $n$ and $m$, provided $\pi/6 < \theta < 5\pi/6$; it is, however, shown that if $n$ and $m$ are real, and such that $n+m-1$, $\frac{1}{2}+m$ are positive, a finite number of terms of the series represents the function approximately when $\theta$ is not subject to the restriction,

$$Q_n^m(\cos\theta) = \sqrt{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left\{\frac{\cos\left(\overline{n+\frac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}}-\frac{1^2-4m^2}{2.\overline{2n+3}}\cdot\frac{\cos\left(\overline{n+\frac{3}{2}}\,\theta+\frac{3\pi}{4}+\frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}}+\ldots\right\},$$

the same remarks holding for this series as for the last one,

$$Q_n^m(\cosh\Psi) = e^{m\pi\iota}\sqrt{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,\frac{e^{-(n+\frac{1}{2})\Psi}}{(2\sinh\Psi)^{\frac{1}{2}}}\left\{1-\frac{1^2-4m^2}{2.\overline{2n+3}}\,\frac{e^{-\Psi}}{2\sinh\Psi}+\frac{1^2-4m^2.\,3^2-4m^2}{2.4.\overline{2n+3}.\overline{n+5}}\,\frac{e^{-2\Psi}}{(2\sinh\Psi)^2}\ldots\right\},$$

where $\cosh\Psi$ must be greater than $3/2\sqrt{2}$.

The following asymptotic values of the functions are obtained:—

$$\frac{\Pi(n)}{\Pi(n+m)}\,P_n^m(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\left\{\left(1-\frac{1-2m^2}{4n}\right)\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)-\frac{1-4m^2}{8n}\cot\theta\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right\},$$

$$\frac{\Pi(n)}{\Pi(n+m)}\,Q_n^m(\cos\theta) = \sqrt{\frac{\pi}{2n\sin\theta}}\left\{\left(1-\frac{1+2m^2}{4n}\right)\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)+\frac{1-4m^2}{8n}\cot\theta\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)\right\},$$

$$\frac{\Pi(n-m)}{\Pi(n)}\,P_n^m(\cosh\Psi) = \frac{1}{\sqrt{n\pi}}\,\frac{e^{-n\Psi}}{\sqrt{1-e^{-2\Psi}}}\left\{1-\frac{3}{8n}+\frac{m^2}{n}+\frac{1-4m^2}{4n}\,\frac{1}{1-e^{-2\Psi}}\right\},$$

$$\frac{\Pi(n)}{\Pi(n-m)}\,Q_n^m(\cosh\Psi) = e^{m\pi\iota}\sqrt{\frac{\pi}{n}}\cdot\frac{e^{-(n+1)\Psi}}{\sqrt{1-e^{-2\Psi}}}\left\{1-\frac{1}{8n}+\frac{m^2}{n}-\frac{1-4m^2}{4n}\,\frac{1}{1-e^{-}}\right\}$$

The following expressions for the functions involving definite integrals along real paths are obtained; these include many known formulæ as special cases :—

$$P_n^m(\mu)-\frac{2}{\pi}e^{-m\pi\iota}\sin m\pi\,.\,Q_n^m(\mu)$$
$$=\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\,\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi(\mu+\sqrt{\mu^2-1}\cos\Psi)^{n-m}\sin^{2m}\Psi\,d\Psi$$

where $n$ is unrestricted, and $m$ must be such that the real part of $m+\frac{1}{2}$ is positive, that of $\mu$ must also be positive; under the same restrictions—

$$P_n^m(\mu)-\frac{2}{\pi}e^{-m\pi\iota}\sin m\pi \,.\, Q_n^m(\mu)$$

$$=\frac{\Pi(n+m)}{\Pi(n-m)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi \frac{\sin^{2m}\Psi}{(\mu+\sqrt{\mu^2-1}\cos\Psi)^{n+m+1}}d\Psi,$$

also

$$P_n^m(\mu)\frac{\Pi(n-m)}{\Pi(n)}(-1)^m \frac{\cos}{\sin} m(\Psi\mp\iota u)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{\frac{\cos}{\sin}m\phi}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\Psi\pm\iota u)\}^{n+1}}d\phi$$

$$P_n^m(\mu)\frac{\Pi(n)}{\Pi(n+m)}\frac{\cos}{\sin}m(\Psi\mp\iota u)=\frac{1}{2\pi}\int_0^{2\pi}\{\mu+\sqrt{\mu^2-1}\cos(\phi-\Psi\pm\iota u)\}^n d\phi$$

where $m$ is a real integer, $n$ is unrestricted, and $u$ is any real positive quantity less than $\frac{1}{2}$ log. mod. $\frac{\mu+1}{\mu-1}$, and the real part of $\mu$ is positive,

$$Q_n^m() = \frac{1}{2^m}e^{m\pi\iota}\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}$$

$$\int_0^\infty(\mu+\sqrt{\mu^2-1}\cosh w)^{-n-m-1}\sinh^{2m}w\,dw$$

where the real parts of $m+\frac{1}{2}$, $n-m+1$ must be positive;

$$Q_n^m(\mu) = e^{m\pi\iota}2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\cos m\pi \,.\, (\mu^2-1)^{-\frac{1}{2}m}$$

$$\int_0^{w_0}(\mu-\sqrt{\mu^2-1}\cosh w)^{n+m}\sinh^{-2m}w\,dw$$

where $w_0=\frac{1}{2}$ log. mod. $\frac{\mu+1}{\mu-1}$, and the real parts of $n+m+1$, $\frac{1}{2}-m$ must be positive;

$$Q_n^m(\mu) = e^{m\pi\iota}\frac{\Pi(n)}{\Pi(n-m)}\int_0^\infty\frac{\cosh mu}{(\mu+\sqrt{\mu^2-1}\cosh u)^{n+1}}du,$$

where the real parts of $n+m+1$ and $n-m+1$ must be positive;

$$Q_n^m(\mu) = e^{m\pi\iota}\frac{\Pi(n+m)}{\Pi(n)}\int_0^{\log_e\sqrt{[(\mu+1)/(\mu-1)]}}\{\mu-\sqrt{\mu^2-1}\cosh u\}^n\cosh mu\,du,$$

where the real part of $n+1$ must be positive.

The following expressions are obtained :—

$$\{\mu+\sqrt{\mu^2-1}\cos(\phi-\Psi\pm\iota u)\}^n$$
$$=\mathrm{P}_n(\mu)+2\sum_{m=1}^{m=\infty}\frac{\Pi(n)}{\Pi(n+m)}\mathrm{P}_n^m(\mu)\cos m\,\overline{\phi-\Psi\pm(\iota u)},$$

where $u<\log$ mod. $\sqrt{\{(\mu+1)/(\mu-1)\}}$; this expansion holds for unrestricted values of $n$, $m$ having all positive integral values;

$$\{\mu+\sqrt{\mu^2-1}\cos(\phi-\Psi\pm\iota u)\}^n$$
$$=\Sigma\frac{\Pi(n)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu)-\frac{2}{\pi}e^{-n\pi\iota}\sin n\pi\,.\,\mathrm{Q}_n^m(\mu)\right\}e^{m\iota(\phi-\Psi-\iota u)},$$

where $u>\log$ mod. $\sqrt{\{(\mu+1)/(\mu-1)\}}$, $n$ is unrestricted and $m$ has the values $n$, $n-1$, $n-2\ldots.$ .

The following generalisations of the well-known expressions of Dirichlet and Mehler for $\mathrm{P}_n(\cos\theta)$ are obtained :—

$$\mathrm{P}_n^{-m}(\cos\theta)=\frac{2}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_0^\theta\frac{\cos(n+\frac{1}{2})\,\phi}{(2\cos\phi-2\cos\theta)^{\frac{1}{2}-m}}\,d\phi,$$

where the real part of $m+\frac{1}{2}$ is positive, and $n$ is unrestricted.

$$\mathrm{P}_n^{-m}(\cos\theta)=\frac{2\sin^{-m}\theta}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\left\{\int_\theta^\pi\frac{\cos[(n+\frac{1}{2})\,\phi-(m+\frac{1}{2})\,\pi]}{(2\cos\theta-2\cos\phi)^{\frac{1}{2}-m}}\,d\phi\right.$$
$$\left.+\cos(n+\tfrac{1}{2}-m)\,\pi\int_0^\infty\frac{e^{-(n+\frac{1}{2})v}}{(2\cos hv+2\cos\theta)^{\frac{1}{2}-m}}\right\},$$

which holds, provided the real parts of $m+\frac{1}{2}$, $n-m+1$ are positive.

Various other definite integral formulæ are deduced which hold under special conditions.

The following recurrent relations are proved for unrestricted values of $n$ and $m$ :—

$$(\mu^2-1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu}=(n-m+1)\,\mathrm{P}_{n+1}^m(\mu)-(n+1)\,\mu\mathrm{P}_n^m(\mu),$$

$$(\mu^2-1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu}=n\mu\mathrm{P}_n^m(\mu)-(n+m)\,\mathrm{P}_{n-1}^m(\mu),$$

$$(2n+1)\,\mu\mathrm{P}_n^m(\mu)-(n-m+1)\mathrm{P}_{n+1}^m(\mu)-(n+m)\,\mathrm{P}_{n-1}^m(\mu)=0,$$

$$\mathrm{P}_n^{m+2}(\mu)+2(m+1)\frac{\mu}{\sqrt{\mu^2-1}}\mathrm{P}_n^{m+1}(\mu)-(n-m)(n+m+1)\mathrm{P}_n^m(\mu)=0,$$

with precisely corresponding formulæ for the Q functions.

The memoir concludes with an examination of the ring functions, and the harmonics for the cone and the bowl; in particular, convergent series are obtained for both the tesseral toroidal functions.