III. "Memoir on the Theory of the Partitions of Numbers. Part I." By Major P. A. MacMahon, R.A., F.R.S.

Received December 31, 1895.

(Abstract.)

The memoir here presented is a natural sequel to my memoirs of 1891, 1893, and 1894, published *in extenso* in the 'Philosophical Transactions.' In fundamental idea it is graphical, resting, on the one hand, upon the method of the memoir on the "Compositions of Numbers," of 1893, and, on the other, upon Sylvester's graphical method, set forth in his "Constructive Theory of Partitions," of 1882, published in vol. 5 of the 'American Journal of Mathematics.'

The memoir is divided into four sections. In § 1 I give new notions concerning the partitions of ordinary unipartite numbers, and show that the theory of the separations of a partition necessitates the consideration of the partitions of multipartite numbers. The two theories proceed in parallel paths. One-to-one correspondence can be established at any point.

In § 2 I am engaged with the graphical representation of unipartite partitions. The graph that, in the memoir of 1893, was employed to denote a principal composition of a bipartite number is shown to be the graph also of a unipartite partition. A new theory of unipartite partitions is evolved with algebraical developments in correspondence.

In § 3 I investigate a similar correspondence between the compositions of tripartite numbers and certain regularised partitions of bipartite numbers. The method is of general application, and indicates a one-to-one correspondence between the compositions of $m+1$-partite numbers and certain regularised partitions of m-partite numbers.

In § 4 I take up the question of the graphical representation of completely regularised multipartite numbers. I follow Sylvester, proceeding from two to three dimensions. Whereas Sylvester employed nodes in a two-dimensional corner, I employ nodes piled up in a three-dimensional corner. Sylvester obtains a two-fold correspondence from the permutations of his axes x, y. I obtain a six-fold correspondence from the permutations of the three axes x, y, z. Even Sylvester's two-dimensional graphs permit of six interpretations when viewed from the three-dimensional standpoint.

I strive to determine the number of graphs appertaining to a given number of nodes, the numbers of nodes along the three axes being restricted in any given manner.

There is no difficulty in forming a generating function which solves

the problem. This function, as formed, is redundant (or, as Sylvester would say, "crude"), as containing terms which do not appertain to the enumeration, and it has been a principal object to obtain condensed or reduced generating functions. It was not difficult to conjecture that if the graphs have not more than two layers of nodes (*i.e.*, not more than two nodes along the axis of z, taken to be perpendicular to the plane of the paper), but be otherwise unrestricted, the reduced generating function is—

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}(1-x^4)^{-2}\ldots(1-x^5)^{-2}\ldots \textit{ ad inf.}$$

Failure to establish this happily led me to send the corresponding crude generating function to Professor Forsyth. He furnished an ingenious solution, which is on the point of appearing in the 'Proceedings of the London Mathematical Society.'

Subsequent to this I obtained a proof, of quite a different character and without reference to the crude form, of the same theorem which will be found in this section.

Conjecturally, the G.F.'s, when the layers are restricted in number to 3, 4, ..., &c., are—

$$(1-x)^{-1}(1-x^2)^{-2}\{(1-x^3)(1-x^4)\ldots \textit{ ad inf.}\}^{-3},$$
$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}\{(1-x^4)(1-x^5)\ldots \textit{ ad inf.}\}^{-4}, \text{ \&c.},$$

and when the graphs are quite unrestricted—

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\ldots(1-x^5)^{-5}\ldots \textit{ ad inf.}$$

Finally, a conjecture is made as to the form when the graphs are restricted in all these dimensions.

These conjectures are slowly being transformed into truths, and I trust to present them as such to the Royal Society as Part 2 of this memoir.

IV. "On some Physical Properties of Argon and Helium." By LORD RAYLEIGH, Sec. R.S. Received January 16, 1896.

Density of Argon.

In our original paper* are described determinations by Prof. Ramsay, of the density of argon prepared with the aid of magnesium. The volume actually weighed was 163 c.c., and the adopted mean result was 19·941, referred to $O_2 = 16$. At that time a satisfactory conclusion as to the density of argon prepared by the oxygen

* Rayleigh and Ramsay, 'Phil. Trans.,' vol. 186, A, pp. 221, 238, 1895.