III. "The Complete System of the Periods of a Hollow Vortex Ring." By H. C. Pocklington, B.A., Fellow of St. John's College, Cambridge. Communicated by Professor Larmor, F.R.S. Received April 23, 1895.

(Abstract.)

The author discusses the stability of a hollow annular vortex in an infinite perfect liquid, and also the effect of an electric charge on the steady motion and the stability of such a vortex. It is known that a hollow vortex ring (without electric charge) is stable for such small deformations as are symmetrical about the axis of symmetry of the ring, and for such as consist in displacement of the axis of the hollow without alteration of the size or shape of its cross section. This investigation shows that, in addition to the fluted and sinuous vibrations above referred to, the vortex is capable of beaded vibrations, in which the hollow is enlarged and contracted at regular intervals along its length, and also of vibration of a more general type, in which the displacement at any moment consists of waves on the surface of the hollow, of which the crests are circles parallel to the axis of the hollow, and the amplitude a sine or cosine of a multiple of the azimuth angle. The periods of these vibrations are found and proved to be real. Since, as is easily seen, any displacement of the surface can be compounded by displacement of the various types here mentioned, the vortex is stable for all displacements of its surface.

When an attempt is made to explain matter as composed of atoms which consist of such vortex rings, a difficulty is found at the outset if the theory is applied to gases. On the kinetic theory of gases, a theory which, over a wide range, gives results in accordance with those of experiment, the energy of an atom varies as the square of its velocity. The energy of a vortex ring, however, decreases as its velocity increases. Thus the single vortex atom theory is likely to yield results that disagree with experimental results when applied to gases. If, however, the electric charge is taken into account, this defect can be, to a certain extent, remedied.

The case where the electricity resides on the surface, supposed conducting, of the hollow of the vortex ring is worked out on the hypothesis that the period of an electrical oscillation is so small that the electricity has at any time its equilibrium distribution. It is found that, in the case of such a ring, however small the charge may be, the velocity of the ring can be decreased, made to vanish, and finally to change sign by decreasing the radius of the ring. At the same time, the energy diminishes, attains a minimum when the velocity of the vortex is zero, and then increases. If therefore the

vortex atom always has nearly the size that corresponds to minimum energy, its energy, neglecting a constant term, varies as the square of its velocity. This relation, however, only holds through a small range. On investigation of the periods of the electrified vortex it is found that it is always unstable for some types of disturbance.

The method employed throughout the investigation is to refer the circumstances of the motion to toroidal co-ordinates. The steady motion of the fluid is then expressed by means of a Stokes' current function, and the disturbances of the steady motion, in the case of vibration, by means of a potential function. The periods of the different oscillations, when real, are given as $2\pi/p$, where p is given by

$$p^2+2pn\frac{\mu}{2\pi e^2}+n(n-1)\left\{1+\frac{E^2}{\pi\rho a^2\mu^2}\right\}\left(\frac{\mu}{2\pi e^2}\right)^2=0$$

in the case of the general oscillation, where there are n waves in the circumference of the hollow;

$$p^2=\left(\frac{\mu}{2\pi e^2}\right)^2\left\{\frac{1}{L'-\frac{1}{2}\gamma_m}+\frac{(L'-\frac{1}{2}\gamma_m)^2+1}{L'-\frac{1}{2}\gamma_m}\frac{E^2}{\pi\rho a^2\mu^2}\right\}$$

in the case of beaded vibrations, where m is the number of enlargements in the circumference of the ring, $L'=\log 8a/e$ and

$$\gamma_m=4\{1+1/3+1/5+\ldots\quad 1/(2m-1)\};$$

$$p^2=\left(\frac{\mu}{2\pi e^2}\right)^2\frac{1}{L'-\frac{1}{2}\gamma_m}\left\{1+\frac{E^2}{\pi\rho a^2\mu^2}\right\}$$

in the case of the pulsations, where μ is the circulation of the ring, a its radius, e the radius of the cross-section of the hollow, ρ the density of the liquid, E the electric charge on the ring. In the case of the general oscillation, in which the disturbance consists of n waves parallel to the axis of the ring, the amplitudes of which vary as cosine or sine of m times the azimuth angle, the value of p must be modified if m is large. The formula for p then is

$$p^2+2pn\frac{\mu}{2\pi e^2}+\left\{n^2+\frac{2mbK'_n(2mb)}{K_n(2mb)}\right.$$
$$\left.+\frac{[2mbK'_n(2mb)]^2+[K_n(2mb)]^2}{2mbK'_n(2mb)\,.\,K_n(2mb)}\frac{E^2}{\pi\rho a^2\mu^2}\right\}\left(\frac{\mu}{2\pi e^2}\right)^2=0.$$