"On the Calculation of the Coefficient of Mutual Induction of a Circle and a Coaxial Helix, and of the Electromagnetic Force between a Helical Current and a Uniform Coaxial Circular Cylindrical Current Sheet." By Professor J. VIRIAMU JONES, F.R.S. Received November 12,—Read December 9, 1897.

§ 1. In measuring electrical resistance by the method of Lorenz we have to determine the coefficient of mutual induction of a helix of wire and the circumference of a rotating circular disc placed coaxially with it, the mean planes of the helix and the disc being coincident. In a paper presented to the Physical Society in November, 1888, I gave a method of calculating this coefficient; but subsequent consideration of the problem in connection with the Lorenz apparatus recently made for the McGill University, Montreal, has led me both to a simplification of the method previously described, and also to a more general solution.

§ 2. If M is the coefficient of mutual induction of any two curves we have

$$M = \iint \frac{\cos \epsilon}{r} ds ds',$$

where r = the distance between two elements ds, ds'; and ϵ = the angle between these elements. Let the equations to the circle and coaxial helix be

$$\left.\begin{aligned} y &= a \cos \theta \\ z &= a \sin \theta \\ x &= 0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} y' &= A \cos \theta' \\ z' &= A \sin \theta' \\ x' &= p\theta' \end{aligned}\right\}$$

Then

$$M = \iint \frac{dxdx' + dydy' + dzdz'}{\surd\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}$$

$$= \int_0^{2\varpi} \int_{\Theta_1}^{\Theta_2} \frac{Aa \cos(\theta-\theta') d\theta d\theta'}{\surd\{A^2 + a^2 - 2Aa \cos(\theta-\theta') + p^2\theta'^2\}}.$$

If we change the variables, putting

$$\left.\begin{aligned}\theta-\theta' &= \phi\\ \theta' &= \phi'\end{aligned}\right\}$$

we find

$$\mathrm{M} = \int_{-\Theta_2}^{2\varpi-\Theta_2}\int_{-\phi}^{\Theta_2} \mathrm{V}d\phi d\phi' + \int_{2\varpi-\Theta_2}^{\Theta_1}\int_{-\phi}^{2\varpi-\phi} \mathrm{V}d\phi d\phi' + \int_{-\Theta_1}^{2\varpi-\Theta_1}\int_{-\Theta_1}^{2\varpi-\phi} \mathrm{V}d\phi d\phi'$$

where $\mathrm{V} = \dfrac{\mathrm{A}a\cos\phi}{\sqrt{(\mathrm{A}^2+a^2-2a\mathrm{A}\cos\phi+p^2\phi'^2)}} = \dfrac{\mathrm{A}a\cos\phi}{\sqrt{(\alpha^2+p^2\phi'^2)}}$

if $$\alpha^2 = \mathrm{A}^2+a^2-2\mathrm{A}a\cos\phi.$$

Now $\displaystyle\int \mathrm{V}d\phi' = \frac{\mathrm{A}a\cos\phi}{p}\log\left(p\phi'+\sqrt{\alpha^2+p^2\phi'^2}\right) = \mathrm{F}(\phi')$, say,

and it may be readily seen by substituting ϕ for $2\varpi-\phi$ in the second and fourth integrals that

$$-\int_{-\Theta_2}^{2\varpi-\Theta_2}\mathrm{F}(-\phi)\,d\phi+\int_{2\varpi-\Theta_2}^{\Theta_1}\mathrm{F}(2\varpi-\phi)\,d\phi-\int_{2\varpi-\Theta_2}^{\Theta_1}\mathrm{F}(-\phi)\,d\phi$$
$$+\int_{-\Theta_1}^{2\varpi-\Theta_1}\mathrm{F}(2\varpi-\phi)\,d\phi = 0.$$

We have, therefore,

$$\mathrm{M} = \int_{-\Theta_2}^{2\varpi-\Theta_2}\mathrm{F}(\Theta_2)\,d\phi - \int_{-\Theta_1}^{2\varpi-\Theta_1}\mathrm{F}(\Theta_1)d\phi;$$

but

$$\int_{-\Theta}^{2\varpi-\Theta}\mathrm{F}(\Theta)\,d\phi = \int_0^{2\varpi}\mathrm{F}(\Theta)\,d\phi, \quad\text{since}\quad \int_{-\Theta}^{0}\mathrm{F}(\Theta)d\phi = \int_{2\varpi-\Theta}^{2\varpi}\mathrm{F}(\Theta)\,d\phi,$$

therefore

$$\mathrm{M} = \int_0^{2\varpi}\frac{\mathrm{A}a\cos\phi}{p}\log\left(p\Theta_2+\sqrt{\alpha^2+p^2\Theta_2^2}\right)d\phi$$
$$-\int_0^{2\varpi}\frac{\mathrm{A}a\cos\phi}{p}\log\left(p\Theta_1+\sqrt{\alpha^2+p^2\Theta_1^2}\right)d\phi \quad \ldots\ldots\ldots\ldots \quad (1).$$

If $\Theta_1 = 0$ and $\Theta_2 = \Theta$,

$$\mathrm{M}_\Theta = \int_0^{2\varpi}\frac{\mathrm{A}a\cos\phi}{p}\log\left(\frac{p\Theta}{\alpha}+\sqrt{1+\frac{p^2\Theta^2}{\alpha^2}}\right)d\phi,$$

which is the coefficient of mutual induction of the circle, and a helix beginning in the plane of the circle of axial length, $p\Theta$.

It is clear that $$M = M_{\Theta_2} - M_{\Theta_1},$$

and we need therefore only consider the expression

$$M_{\Theta} = \int_0^{2\varpi} \frac{Aa\cos\phi}{p} \log\left(\frac{x}{\alpha} + \sqrt{1+\frac{x^2}{\alpha^2}}\right) d\phi \quad \ldots\ldots \quad (2),$$

where $x = p\Theta =$ the axial length of the helix, reckoned from the plane of the circle.

We may now proceed in two ways—either by expanding the logarithmic expression in powers of x/a, which leads to a series of limited application since it is convergent only so long as $x < A - a$; or by integration by parts which leads to an expression applicable for all values of x.

§ 3. The first method I developed in the paper above mentioned. We have

$$M_{\Theta} = \frac{Aa}{p}\int_0^{2\varpi} \Sigma(-1)^m \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m} \frac{1}{2m+1} \frac{x^{2m+1}}{\alpha^{2m+1}} \cos\phi d\phi$$

$$= \frac{Aa}{p} \Sigma(-1)^m \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m} \cdot \frac{1}{2m+1} \cdot x^{2m+1} \int_0^{2\varpi} \frac{\cos\phi d\phi}{\alpha^{2m+1}}$$

$$= 4\frac{Aa}{p} \Sigma(-1)^{m+1} \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m} \frac{1}{2m+1} \left(\frac{x}{A+a}\right)^{2m+1} P_m$$

$$= \Theta(A+a)c^2 \Sigma(-1)^{m+1} \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m} \frac{1}{2m+1} g^{2m} P_m \ldots \quad (3)$$

where $$c = \frac{2\sqrt{Aa}}{A+a}, \qquad g = \frac{x}{A+a},$$

$$P_m = \int_0^{\frac{\varpi}{2}} \frac{\cos 2\theta d\theta}{(1-c^2\sin^2\theta)^{(2m+1)/2}}.$$

Let $$Q_m = \int_0^{\frac{\varpi}{2}} \frac{d\theta}{(1-c^2\sin^2\theta)^{(2m+1)/2}}.$$

The following properties of these elliptic integrals are perhaps worthy of notice :—

$$P_m = \left(1-\frac{2}{c^2}\right) Q_m + \frac{2}{c^2} Q_{m-1} \quad \ldots\ldots\ldots\ldots \quad (\text{i}),$$

$$Q_m = Q_{m-1} + \frac{c}{2m-1} \dot{Q}_{m-1} \quad \ldots\ldots\ldots\ldots \quad (\text{ii}),$$

$$P_m = P_{m-1} + \frac{c}{2m-1}\dot{P}_{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

$$(2m+1)c'^2 Q_{m+1} = 2m(1+c'^2)Q_m - (2m-1)Q_{m-1} \quad \ldots\ldots\ldots \text{(iv)},$$

$$(2m+1)c'^2 P_{m+1} = 2m(1+c'^2)P_m - \frac{(2m-3)(2m+1)}{2m-1}P_{m-1} \quad .. \text{(v)},$$

$$c'^2\dot{Q}_m = \dot{Q}_{m-1} + cQ_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)},$$

$$cc'^2\dot{Q}_m = (2m - c'^2)Q_m - (2m-1)Q_{m-1} \quad \ldots\ldots\ldots \text{(vii)},$$

$$c'^2\dot{Q}_m = \left(1 + c'^2 + \frac{c^2}{2m-1}\right)\dot{Q}_{m-1} - \dot{Q}_{m-2} \quad \ldots\ldots\ldots \text{(viii)},$$

where $c'^2 = 1 - c^2$, and the dotting of a function denotes differentiation with regard to c.

It will be observed that Q_0 and Q_{-1} are respectively the complete elliptic integrals (F and E) of the first and second kinds with regard to modulus c.

§ 4. In equation (3) put

$$K_m = -\frac{1.3.5\ldots.(2m-1)}{2.4.6\ldots.2m} g^{2m}P_m\,,$$

so that $$M_\Theta = \Theta(A+a)c^2\Sigma(-1)^m K_m.$$

Then we can find by a double application of (v) a relation between K_{m+1}, K_m, and K_{m-1}, viz. :—

$$K_{m+1} = \frac{2m(2m+1)}{(2m+2)(2m+3)}d^2\left\{K_m - \frac{(2m-1)(2m-3)}{2m\,.\,2m}e^2K_{m-1}\right\} .. \quad (4)$$

where $$d^2 = \frac{1+c'^2}{c'^2}g^2 \quad \text{and} \quad e^2 = \frac{1}{1+c'^2}g^2\,.$$

This formula renders the calculation of successive terms of the series sufficiently easy.

§ 5. Hence to find M_Θ, given A, a, and x, we have to calculate the following quantities in order :—

$$c = \frac{2\sqrt{Aa}}{A+a}, \quad \text{and} \quad c' = \sqrt{(1-c^2)}\,,$$

$$F(c) = \int_0^{\frac{\varpi}{2}} \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}}, \quad \text{and} \quad E(c) = \int_0^{\frac{\varpi}{2}} \sqrt{(1-c^2\sin^2\theta)}\,d\theta\,,$$

$$K_0 = -P_0 = -\left\{\left(1-\frac{2}{c^2}\right)F + \frac{2}{c^2}E\right\}\,,$$

$$g^2 = \left(\frac{x}{A+a}\right)^2,$$

$$K_1 = -\tfrac{1}{2} \cdot \tfrac{1}{3} g^2 P_1 = -\tfrac{1}{2} \cdot \tfrac{1}{3} g^2 \left\{ \left(1 - \frac{2}{c^2}\right) \frac{E}{c'^2} + \frac{2}{c^2} F \right\},$$

$$d^2 = \frac{1+c'^2}{c'^2} g^2, \qquad \text{and} \qquad e^2 = \frac{1}{1+c'^2} g^2,$$

K_2, K_3, K_4, &c., by successive applications of (4),

$\Sigma(-1)^m K_m$,

and finally $M_\Theta = \Theta(A+a)c^2\Sigma(-1)^m K_m$.

§ 6. An example may be useful in showing the magnitudes of the various quantities concerned.

If

$$\begin{aligned} 2A &= 21{\cdot}02673 \text{ inches}, \\ 2a &= 13{\cdot}01997 \text{ inches}, \\ 2x &= 5{\cdot}02480 \text{ inches}, \\ \Theta &= 201\,\varpi, \end{aligned}$$

then

$$\begin{aligned} c &= 0{\cdot}9719540, \\ c' &= 0{\cdot}2351708, \\ F &= 2{\cdot}8598352, \\ E &= 1{\cdot}0655716, \\ K_0 &= 0{\cdot}9387751, \\ g^2 &= 0{\cdot}02178146, \\ K_1 &= 0{\cdot}0561543, \\ d^2 &= 0{\cdot}4156218, \\ e^2 &= 0{\cdot}0206400, \\ K_2 &= 0{\cdot}0076057, \\ K_3 &= 0{\cdot}0014623, \\ K_4 &= 0{\cdot}0003387, \\ K_5 &= 0{\cdot}0000876, \\ K_6 &= 0{\cdot}0000244, \\ K_7 &= 0{\cdot}0000071, \\ \Sigma(-1)^m K_m &= 0{\cdot}8890325, \\ M_\Theta &= 9028{\cdot}182 \text{ inches} = 22931{\cdot}166 \text{ cm.} \end{aligned}$$

If the circle is in the mean plane of a helix of axial length $2x$, the coefficient of mutual induction will be $2M_\Theta$, or in case of the above dimensions,

$$M = 2M_\Theta = 18056{\cdot}364 \text{ inches} = 45862{\cdot}332 \text{ cm.}$$

The value of M_Θ given above was obtained in 1896 by

Mr. Rhodes under the direction of Prof. Ayrton in the Physical Laboratory of the Central Institution, in which the Lorenz apparatus of the McGill University was tested by Prof. Ayrton and myself (Appendix to the Report of the Electrical Standards Committee of the British Association, 1897). The calculation was made by the somewhat laborious method indicated in my paper "On the Determination of the Specific Resistance of Mercury in Absolute Measure."* It was checked by Mr. Mather, and subsequently I calculated $M_{\odot}$ afresh by the method given here.

§ 7. It is important in practice to determine the change in $M_{\odot}$ consequent on small changes in A, a, and x, both for the calculation of the effect on $M_{\odot}$ of small errors of measurement, and because from time to time the disc of any Lorenz apparatus needs to be re-ground in place, and possibly the coil, owing to a breakdown in insulation, may sometimes need to be re-wound.

Let $$q = \frac{A}{M_{\odot}}\frac{dM_{\odot}}{dA}, \qquad r = \frac{a}{M_{\odot}}\frac{dM_{\odot}}{da}, \qquad s = \frac{x}{M_{\odot}}\frac{dM_{\odot}}{dx},$$

then $$\frac{dM_{\odot}}{M_{\odot}} = q\frac{dA}{A} + r\frac{da}{a} + s\frac{dx}{x}.$$

It may be shown by direct differentiation using relation (iii) between the P functions and their differential coefficients that

$$\left.\begin{aligned} s &= 2T/W \\ q &= \frac{1-s}{2} + \frac{T+2V}{deW} \\ r &= \frac{1-s}{2} - \frac{T+2V}{deW} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (5),$$

where $$W = \Sigma\,(-1)^m K_m,$$

$$T = \Sigma\,(-1)^m m K_m,$$

$$V = \Sigma\,(-1)^m \frac{m}{2m-1} K_m,$$

and d, e have their former significations. W has already been found in calculating $M_{\odot}$; and T and V are easily calculable from the known values of K_0, K_1, K_2, K_3, &c.

Since $M_{\odot}$ is a homogeneous function of the first degree in A, a, x it follows that $q + r + s = 1$, as is obvious in the formulæ above given.

* 'Phil. Trans.,' A (1891), p. 21.

§ 8. For the values of A, a, x given in § 6 we find

$$\begin{aligned} W &= 0{\cdot}8890325, \\ T &= -0{\cdot}0443163, \\ 2V &= -0{\cdot}1036138, \\ \log de &= \bar{2}{\cdot}9667036, \\ s &= -0{\cdot}0997, \\ q &= -1{\cdot}2467, \\ r &= +2{\cdot}3464, \end{aligned}$$

or $$dM_{\Theta}/M_{\Theta} = -1{\cdot}2467dA/A + 2{\cdot}3464da/a - 0{\cdot}0997dx/x.$$

Let us suppose that after re-winding the coil mentioned in § 6, and regrinding the disc, the diameters become

$$2A' = 21{\cdot}02459, \qquad 2a' = 13{\cdot}01499.$$

Then $$rda/a = -0{\cdot}000899,$$
$$qdA/A = +0{\cdot}000127,$$

and $dM_{\Theta}/M_{\Theta} = -0{\cdot}000772$ or $dM_{\Theta} = 6{\cdot}97$ inches $= 17{\cdot}70$ cm.

Therefore $M'_{\Theta} = 9021{\cdot}21$ inches $= 22913{\cdot}47$ cm.

§ 9. I now pass on to the second and more general method of dealing with the expression—

$$M_{\Theta} = \frac{Aa}{p}\int_0^{2\varpi} \cos\phi \log\left(\frac{x}{\alpha} + \sqrt{1+\frac{x^2}{\alpha^2}}\right) d\phi.$$

Integrating by parts we have

$$M_{\Theta} = \left[\frac{Aa\sin\phi}{p}\log\left(\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right)\right]_0^{2\varpi} + \frac{A^2a^2x}{p}\int_0^{2\varpi}\frac{\sin^2\phi d\phi}{\alpha^2\sqrt{(\alpha^2+x^2)}}$$

$$= A^2a^2\Theta\int_0^{2\varpi}\frac{\sin^2\phi d\phi}{\alpha^2\sqrt{(\alpha^2+x^2)}}$$

$$= \frac{16A^2a^2\Theta}{(A+a)^2\sqrt{\{(A+a)^2+x^2\}}}\int_0^{\frac{\varpi}{2}}\frac{\sin^2\psi\cos^2\psi d\psi}{(1-c^2\sin^2\psi)\sqrt{(1-k^2\sin^2\psi)}},$$

where $$c^2 = \frac{4Aa}{(A+a)^2}, \quad \text{and} \quad k^2 = \frac{4Aa}{(A+a)^2+x^2}.$$

Hence by simple changes we have

$$M_{\Theta} = \Theta(A+a)ck\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(F-\Pi)\right\} \cdots\cdots\cdots \quad (6),$$

where F and E are complete elliptic integrals of the first and second kinds to modulus k,

and $$\Pi = \int_0^{\frac{\varpi}{2}} \frac{d\psi}{(1-c^2\sin^2\psi)\sqrt{(1-k^2\sin^2\psi)}}.$$

This expression for $M_\odot$ is applicable for all values of x from 0 to ∞. The elliptic integral of the third kind Π is expressible by Legendre's formula in terms of complete and incomplete integrals of the first and second kind. Thus, if we put

$$-c^2 = -1+k'^2\sin^2\beta,$$

or $$\sin\beta = c'/k',$$

we have (*v.* Cayley, "Elliptic Integrals," § 183)

$$\frac{k'^2\sin\beta\cos\beta}{c}(F-\Pi)$$

$$= -\frac{\varpi}{2} - F(k)F(k',\beta) + E(k)F(k',\beta) + F(k)E(k',\beta).$$

These elliptic integrals, complete and incomplete, may be conveniently calculated by successive quadric transformations, as shown in Cayley, Chap. XIII.

§ 10. Taking for the diameters of helix and circle, the values given in § 8, *viz.*,

$$2A' = 21{\cdot}02459, \qquad 2a' = 13{\cdot}01499,$$

and as before $$2x = 5{\cdot}02480,$$

Professor Ayrton and I made the calculation of $M_\odot$ by equation (6) with the following results:—

$$c = 0{\cdot}9719222,$$
$$k = 0{\cdot}9615024,$$
$$k' = 0{\cdot}2747959,$$
$$L\sin\beta = 9{\cdot}9326156,$$
$$F(k) = 2{\cdot}7109750,$$
$$E(k) = 1{\cdot}0840174,$$
$$F(k',\beta) = 1{\cdot}0393881,$$
$$E(k',\beta) = 1{\cdot}0168643,$$
$$\frac{k'^2\sin\beta\cos\beta}{c}(F-\Pi) = -0{\cdot}5051433,$$
$$\frac{c'^2}{c^2}(F-\Pi) = -0{\cdot}8616240,$$
$$\frac{F-E}{k^2} = 1{\cdot}7598494,$$
$$M_\odot = 22913{\cdot}59.$$

This result afforded a sufficiently satisfactory proof of the accuracy of the calculations, seeing that the result in § 8 was obtained by the application of equations (3) and (5), and that it approximated so closely to the result just given. The approximation would be still closer if we had taken account of the neglected terms of the series, K_8 and K_9.

§ 11. To find corresponding general expressions for the rates of variation of M_Θ with A, a, and x it is simplest to revert to the expression

$$M_\Theta = \frac{Aa}{p}\int_0^{2\varpi} \cos\phi \log\left(\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right)d\phi .$$

We have $$\frac{dM_\Theta}{dx} = -\frac{M_\Theta}{x}+\frac{Aa\Theta}{x}\int_0^{2\varpi}\frac{\cos\phi\, d\phi}{\sqrt{(a^2+x^2)}}$$

(putting $p = x/\Theta$),

$$= -\frac{M_\Theta}{x}+\frac{\Theta(A+a)\,ck}{x}\,P_0(k),$$

$$\frac{dM_\Theta}{dA} = \frac{M_\Theta}{A}-Aa\Theta\int_0^{2\varpi}\cos\phi\frac{A-a\cos\phi}{a^2\sqrt{(a^2+x^2)}}\,d\phi,$$

$$= -Aa\Theta\int_0^{2\varpi}\frac{A\cos\phi-a}{a^2\sqrt{(a^2+x^2)}}\,d\phi,$$

$$= \Theta ck\left\{F+\frac{A-a}{2a}(F-\Pi)\right\},$$

where F and Π have the same significations as in equation (6).

Similarly $$\frac{dM_\Theta}{da} = \Theta ck\left\{F-\frac{A-a}{2A}(F-\Pi)\right\},$$

Hence
$$\left.\begin{aligned} q &= \frac{\Theta ck}{M_\Theta}A\left\{F+\frac{A-a}{2a}(F-\Pi)\right\}\\ r &= \frac{\Theta ck}{M_\Theta}a\left\{F-\frac{A-a}{2a}(F-\Pi)\right\}\\ s &= -1-\frac{\Theta ck}{M_\Theta}(A+a)\left\{\left(1-\frac{2}{k^2}\right)F+\frac{2}{k^2}E\right\}\end{aligned}\right\} \quad \ldots (7).$$

It is readily deducible from equations (7) that

$$q+r+s = 1.$$

When M_Θ has been calculated by (6) it is a simple matter to calculate q, r, s by the above equations.

Let $\dfrac{M_\Theta}{\Theta (A+a) ck} = Z$. Then equations (7) may be expressed thus:—

$$\left.\begin{aligned} q &= \frac{A}{2aZ}(F - c'\Pi) \\ r &= \frac{a}{2AZ}(F + c'\Pi) \\ s &= -1 - P_0(k)/Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (8).$$

For the values of $2A'$, $2a'$, $2x$ taken in § 10 we have

$$\begin{aligned} \Pi &= \quad 17{\cdot}411370, \\ \Pi c' &= \quad 4{\cdot}096932, \\ F - \Pi c' &= -1{\cdot}385957, \\ F + \Pi c' &= \quad 6{\cdot}807907, \\ Z &= \quad 0{\cdot}8982254, \\ P_0(k) &= -0{\cdot}8087238, \\ q &= -1{\cdot}24629, \\ r &= +2{\cdot}34593, \\ s &= -0{\cdot}09964. \end{aligned}$$

On the Potential Energy of a Circular Current and a Uniform Coaxial Circular Cylindrical Current Sheet.

§ 12. The current lines in the sheet are circles in planes at right angles to the axis.

Let the circle have its centre at the origin, and let its equations be

$$\left.\begin{aligned} y &= a\cos\theta \\ z &= a\sin\theta \end{aligned}\right\}$$

and let the equations to the coaxial cylindrical sheet be

$$\left.\begin{aligned} y' &= A\cos\theta' \\ z' &= A\sin\theta' \end{aligned}\right\}$$

the plane ends of the cylinder being determined by the equations $x = x_1$, $x = x_2$.

Let the current in the circle be γ_c, and the current per unit length of the cylindrical sheet γ; and let the potential energy of the circular current and the current sheet be M'.

Then
$$\mathrm{M}' = \gamma_c\gamma\int_{x_1}^{x_2} dx\int_0^{2\varpi} d\theta\int_0^{2\varpi} d\theta' \frac{\mathrm{A}a\cos(\theta-\theta')}{\sqrt{\{\mathrm{A}^2+a^2-2\mathrm{A}a\cos(\theta-\theta')+x^2\}}}$$
$$= 2\varpi\gamma_c\gamma\int_0^{2\varpi} d\phi\int_{x_1}^{x_2} dx\,\frac{\mathrm{A}a\cos\phi}{\sqrt{(\alpha^2+x^2)}}$$

or
$$\mathrm{M}' = 2\varpi\gamma_c\gamma\mathrm{A}a\left[\int_0^{2\varpi}\cos\phi\log\left(\frac{x_2}{\alpha}+\sqrt{1+\frac{x_2^2}{\alpha^2}}\right)d\phi - \int_0^{2\varpi}\cos\phi\log\left(\frac{x_1}{\alpha}+\sqrt{1+\frac{x_1^2}{\alpha^2}}\right)d\phi\right]$$

For a circle and coaxial helix we find the potential energy by multiplying their coefficient of mutual induction by the product of γ_c, γ_h, the currents in the circle and helix respectively. Taking the expression for the coefficient of mutual induction

$$\mathrm{M} = \mathrm{M}_{\Theta_2} - \mathrm{M}_{\Theta_1},$$

M_{Θ_2} and M_{Θ_1} being expressed by equation (2), and noting that $\gamma_h = 2\varpi p\gamma$ where γ is the current per unit length of the helix measured parallel to the axis, we see that the potential energy of the circle and coaxial helix is identically the same as the potential energy of the circle and a uniform coaxial circular cylindrical current sheet of the same radial and axial dimensions as the helix, if the currents per unit length in helix and sheet be the same.*

On the Potential Energy of a Helical Current and a Uniform Coaxial Circular Cylindrical Current Sheet.

§ 13. If we attempt to integrate the general expression for the coefficient of mutual induction in the case of two coaxial helices we are brought face to face with functions which indeed deserve the attention of mathematicians, but do not at present lend themselves to calculation.

For many practical purposes, however, it will be equally useful to calculate the potential energy of a helix and a coaxial uniform circular cylindrical current sheet, or of two coaxial uniform circular cylindrical current sheets; and this potential energy is expressible in terms of elliptic integrals.

Let the equations to a helical current be as before,

$$\left.\begin{aligned} y' &= \mathrm{A}\cos\theta' \\ z' &= \mathrm{A}\sin\theta' \\ x' &= p\theta' \end{aligned}\right\},$$

* *I.e.*, if the current across a generating line of the sheet equal to the pitch of the helix is equal to the helical current.—J. V. J., April 21, 1898.

the limits of x' being x'_1 and x'_2, so that the axial length of the helix is $x'_2 - x'_1$.

Let the equations to a coaxial circular cylindrical current sheet be

$$y = a\cos\theta, \qquad z = a\sin\theta,$$

its ends being at distances x_1, x_2 from the origin.

Let γ_h be the current in the helix, and let γ be the current per unit length of the uniform current sheet. Then if M$'$ is the potential energy of the helical current and the current sheet we have

$$\mathrm{M}' = \int_{x_1}^{x_2}\gamma_h\gamma dx\int_0^{2\varpi}\frac{\mathrm{A}a\cos\phi}{p}\left[\log\left(\frac{x'_2-x}{a}+\sqrt{1+\frac{(x'_2-x)^2}{a^2}}\right)\right.$$
$$\left.-\log\left(\frac{x'_1-x}{a}+\sqrt{1+\frac{(x'_1-x)^2}{a^2}}\right)\right]d\phi$$

$$= \frac{\mathrm{A}a}{p}\gamma_h\gamma\int_0^{2\varpi}\cos\phi d\phi\,[f(x'_2-x_1)-f(x'_2-x_2)+f(x'_1-x_2)-f(x'_1-x_1)]$$

where $$f(z) \equiv z\log\left(\frac{z}{a}+\sqrt{1+\frac{z^2}{a^2}}\right)-\sqrt{a^2+z^2}.$$

The integral $\int_0^{2\varpi}\cos\phi f(z)d\phi$ is easily reducible to elliptic integrals of standard form.

It is to be noticed that $f(z) = f(-z)$.

If the axial length of the helix $= 2l$, and the axial length of the cylindrical sheet $= 2m$, and if $\bar{x}$ equals the distance between their mean planes, we have

$$\mathrm{M}' = \frac{\mathrm{A}a}{p}\gamma_h\gamma\int_0^{2\varpi}\cos\phi d\phi\,[f(\bar{x}+l+m)-f(\bar{x}+l-m)$$
$$+f(\bar{x}-l-m)-f(\bar{x}-l+m)] \quad \ldots \quad (9).$$

Similarly for two coaxial cylindrical current sheets we have

$$\mathrm{M}' = 2\varpi\mathrm{A}a\gamma\gamma'\int_0^{2\varpi}\cos\phi d\phi\,[f(\bar{x}+l+m)-f(\bar{x}+l-m)$$
$$+f(\bar{x}-l-m)-f(\bar{x}-l+m)] \ldots \quad (10),$$

where γ, γ' are the currents per unit length in the sheets.

§ 14. To find the force between a helical current and a coaxial circular cylindrical uniform current sheet we have to differentiate the potential energy as expressed in equation (9) with regard to x.

We have, if X is the force,

$$X = \frac{dM'}{d\bar{x}} = \gamma_h\gamma\frac{Aa}{p}\int_0^{2\varpi}\cos\phi\, d\phi\,[F(\bar{x}+l+m)-F(\bar{x}+l-m)$$
$$+F(\bar{x}-l-m)-F(\bar{x}-l+m)],$$

where $$F(z) = f'(z) = \log\left(\frac{z}{\alpha}+\sqrt{1+\frac{z^2}{\alpha^2}}\right),$$

or by equation (2)

$$X = \gamma_h\gamma(M_2-M_1) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (11),$$

where M_2 = coefficient of mutual induction of the helix and one of the circular ends of the sheet,

and M_1 = coefficient of mutual induction of the helix and the other circular end of the sheet.

M_2 and M_1 may be calculated as described in the previous articles of this paper.

§ 15. Equation (11) is clearly a particular case of a more general theorem.

Take any cylindrical sheet developed by the rectilinear translation of a given closed curve, and let the sheet be the seat of a uniformly distributed current, the current lines being successive positions of the given curve as by its translation it developes the sheet. Let the current per unit length of the sheet be γ. Further, let M_1 be the coefficient of mutual induction of the given curve in its first position and any second fixed curve, and M_2 the coefficient of mutual induction of the given curve in its last position and the second curve; and let γ_2 be the current in the second curve.

Then the force between the current sheet and the second curve resolved parallel to the direction of translation of the given curve as it developes the sheet is given by the formula—

$$F = \gamma_2\gamma(M_2-M_1).$$

For let the direction of translation be taken as the axis of x; and let M_x be the coefficient of mutual induction of the given curve in any intermediate position defined by the co-ordinate x, *i.e.*, of an element of the current sheet, and the second curve. Then the force resolved parallel to x between the element of the current sheet and the second curve

$$= \gamma_2 \cdot \gamma dx \cdot dM_x/dx,$$

and the total force so resolved between the current sheet and the second curve

$$= \int_{x_1}^{x_2} \gamma_2 \cdot \gamma dx \cdot d\text{M}_x/dx = \gamma_2\gamma\,(\text{M}_2-\text{M}_1).*$$

As special cases, we have equation (11) reducing the calculation of the force between a circular cylindrical uniform current sheet and a coaxial helical current to the calculation of the coefficients of mutual induction of the helix and the circular ends of the sheet; and the simpler case of the force between a circular cylindrical uniform current sheet and a circular current, which is obtained from the calculations of the coefficients of mutual induction of the circle and the circular ends of the sheet.

I hope that equation (11) may be of service in the accurate calculation of the constants of current weighing apparatus. My attention was drawn to the matter from this point of view in consequence of the Report of the Electrical Standards Committee of the British Association made at Toronto, in which mention is made of the importance of re-determining the ampere.

March 31, 1898.

The LORD LISTER, F.R.C.S., D.C.L., President, in the Chair.

Preliminary communications upon the results of the recent Solar Eclipse were made by the following members of the expeditions:—

The Astronomer Royal, Sir J. Norman Lockyer, K.C.B., Professor H. H. Turner, Dr. R. Copeland, and Captain E. H. Hills, R.E., and Mr. H. F. Newall.

The Society adjourned over the Easter Recess to Thursday, April 28.

* Professor A. Gray has pointed out to me that this result may be deduced from the consideration that the removal of an element from one end of the sheet to the other is equivalent to a small motion of the sheet parallel to its generating lines.—J. V. J., April 21, 1898.