"Memoir on the Theory of the Partitions of Numbers. Part II."

By Major P. A. MacMahon, R.A., D.Sc., F.R.S. Received November 21,—Read November 24, 1898.

(Abstract.)

Introduction.

The subject of the partition of numbers, for its proper development, requires treatment in a new and more comprehensive manner. The subject matter of the theory needs enlargement. This will be found to be a necessary consequence of the new method of regarding a partition that is here brought into prominence.

Let an integer n be broken up into any number of integers

$$\alpha_1,\ \alpha_2,\ \alpha_3, \ldots\ldots \alpha_s,$$

and we ascribe the conditions

$$\alpha_1 \eqslantgtr \alpha_2 \eqslantgtr \alpha_3 - \ldots\ldots \eqslantgtr \alpha_s,$$

the succession

$$\alpha_1,\ \alpha_2,\ \alpha_3 \ldots\ldots \alpha_s$$

is what is known as a partition of n.

There are $s-1$ conditions

$$\alpha_1 \eqslantgtr \alpha_2,\ \alpha_2 \eqslantgtr \alpha_3 \ldots\ldots \alpha_{s-1} \eqslantgtr \alpha_s,$$

to which we may add

$$\alpha_s \eqslantgtr 0,$$

if the integers be all of them positive (or zero).

For the present all the integers are restricted to the positive or zero by hypothesis, so that this last-written condition will not be further attended to.

If, on the other hand, the conditions be

$$\alpha_1 \gtreqless \alpha_2 \gtreqless \alpha_3 \ldots\ldots \gtreqless \alpha_s$$

no order of magnitude is supposed to exist between the successive parts, and we obtain what has been termed a "composition" of the integer n.

Various other systems of partitions into s parts may be brought under view, because between two consecutive parts we may place either of the seven symbols

$$>,\ =,\ <,\ \geqq,\ \leqq,\ \gtrless,\ \gtreqless.$$

We thus obtain 7^{s-1} different sets of conditions that may be assigned; these are not all essentially different, and in many cases they overlap.

For the moment I concentrate attention upon the symbol

$$\geqq$$

and remark that the $s-1$ conditions which involve this symbol, set forth above, constitute one set of a larger class of sets which involve the symbol. We may have the single condition

$$A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + A_3^{(1)}\alpha_3 + \ldots\ldots + A_s^{(1)}\alpha_s \geqq 0,$$

wherein $A_1, A_2, A_3 \ldots\ldots A_s$ are integers $+$, zero, or $-$, of which at least one must be positive, or we may have the set of conditions

$$\left.\begin{array}{l} A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + A_3^{(1)}\alpha_3 + \ldots\ldots + A_s^{(1)}\alpha_s \geqq 0 \\ A_1^{(2)}\alpha_1 + A_2^{(2)}\alpha_2 + A_3^{(2)}\alpha_3 + \ldots\ldots + A_s^{(2)}\alpha_s \geqq 0 \\ A_1^{(3)}\alpha_1 + A_2^{(3)}\alpha_2 + A_3^{(3)}\alpha_3 + \ldots\ldots + A_s^{(3)}\alpha_s \geqq 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_1^{(r)}\alpha_1 + A_2^{(r)}\alpha_2 + A_3^{(r)}\alpha_3 + \ldots\ldots + A_s^{(r)}\alpha_s \geqq 0 \end{array}\right\}$$

as the definition of the partitions considered.

If the symbol be $=$ instead of $\geqq$ the solution of the equations falls into the province of linear Diophantine analysis. The problem before us may be regarded as being one of linear partition analysis. There is much in common between the two theories; the problems may be treated by somewhat similar methods, and lead to results of the same general character.

The partition analysis of degree higher than the first, like the Diophantine, is of a more recondite nature, and is left for the present out of consideration.

I treat the partition conditions by the method of generating functions. I seek the summation

$$\Sigma\, X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \ldots\ldots X_s^{\alpha_s}$$

for every set of values (integers)

$$\alpha_1,\ \alpha_2,\ \alpha_3, \ldots\ldots \alpha_s$$

which satisfy the assigned conditions.

It appears that there are, in every case, a finite number of ground or fundamental solutions of the conditions, viz. :—

$$\begin{array}{l} \alpha_1^{(1)},\ \alpha_2^{(1)},\ \alpha_3^{(1)} \ldots\ldots \alpha_s^{(1)} \\ \alpha_1^{(2)},\ \alpha_2^{(2)},\ \alpha_3^{(2)} \ldots\ldots \alpha_s^{(2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1^{(m)},\ \alpha_2^{(m)},\ \alpha_3^{(m)} \ldots\ldots \alpha_s^{(m)} \end{array}$$

such that every solution

$$\alpha_1,\ \alpha_2,\ \alpha_3, \ldots\ldots \alpha_s$$

is such that

$$\begin{aligned}
\alpha_1 &= \lambda_1\alpha_1^{(1)} + \lambda_2\alpha_1^{(2)} \ldots\ldots + \lambda_m\alpha_1^{(m)} \\
\alpha_2 &= \lambda_1\alpha_2^{(1)} + \lambda_2\alpha_2^{(2)} \ldots\ldots + \lambda_m\alpha_2^{(m)} \\
\alpha_3 &= \lambda_1\alpha_3^{(1)} + \lambda_2\alpha_3^{(2)} \ldots\ldots + \lambda_m\alpha_3^{(m)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_s &= \lambda_1\alpha_s^{(1)} + \lambda_2\alpha_s^{(2)} \ldots\ldots + \lambda_m\alpha_s^{(m)}
\end{aligned}$$

$\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ being positive integers.

This arises from the fact that every term

$$X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \ldots\ldots X_s^{\alpha_s}$$

of the summation is found to be expressible as a product

$$\begin{aligned}
&\left\{ X_1^{\alpha_1^{(1)}} X_2^{\alpha_2^{(1)}} X_3^{\alpha_3^{(1)}} \ldots\ldots X_s^{\alpha_s^{(1)}} \right\}^{\lambda_1} \\
\times &\left\{ X_1^{\alpha_1^{(2)}} X_2^{\alpha_2^{(2)}} X_3^{\alpha_3^{(2)}} \ldots\ldots X_s^{\alpha_s^{(2)}} \right\}^{\lambda_2} \\
\times &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\times &\left\{ X_1^{\alpha_1^{(m)}} X_2^{\alpha_2^{(m)}} X_3^{\alpha_3^{(m)}} \ldots\ldots X_s^{\alpha_s^{(m)}} \right\}^{\lambda_m}
\end{aligned}$$

Denoting this product by

$$P_1^{\lambda_1} P_2^{\lambda_2} \ldots\ldots P_m^{\lambda_m}$$

the generating function assumes the form

$$\frac{1 - (Q_1^{(1)} + Q_1^{(2)} + Q_1^{(3)} + \ldots) + (Q_2^{(1)} + Q_2^{(2)} + Q_2^{(3)} + \ldots) - (Q_3^{(1)} + \ldots) + \ldots}{(1 - P_1)(1 - P_2) \ldots\ldots (1 - P_m)}$$

wherein the denominator indicates the ground solutions and the numerator the simple and compound syzygies which unite them.

The terms

$Q_1^{(1)},\ Q_1^{(2)},\ Q_1^{(3)} \ldots\ldots$ denote first syzygies

$Q_2^{(1)},\ Q_2^{(2)},\ Q_2^{(3)} \ldots\ldots$,, second ,,

$Q_3^{(1)},\ Q_3^{(2)},\ Q_3^{(3)} \ldots\ldots$,, third ,,

....

The reader will note the striking analogy with the generating functions of the theory of invariants.

Similar results are obtained as solutions of linear Diophantine equations.

The generating functions under view are *real* in the sense of Cayley and Sylvester. Enumerating generating functions of various kinds are obtained by assigning equalities between the suffixed capitals

$$X_1,\ X_2, \ldots\ldots X_s.$$

Putting, *e.g.*, $\quad X_1 = X_2 = \ldots\ldots = X_s = x,$

we obtain the function which enumerates by the coefficient of x^n, in the ascending expansion, the numbers of solutions for which

$$\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_3 = n.$$

It will be gathered that the note of the following investigation is the importation of the idea that the solution of any system of equations of the form

$$A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + \ldots\ldots + A_s\alpha_s \geq 0$$

(all the quantities involved being integers) is a problem of partition analysis, and that the theory proceeds *pari passu* with that of the linear Diophantine analysis.

"On the Boiling Point of Liquid Hydrogen under Reduced Pressure." By JAMES DEWAR, M.A., LL.D., F.R.S. Received and Read December 15, 1898.

The June number of the 'Proceedings of the Chemical Society' contains a paper by the author on "The Boiling Point and Density of Liquid Hydrogen." A resistance thermometer made of fine platinum wire, called No. 7 Thermometer, was used in the investigation. It had been carefully calibrated, and gave the following resistances at different temperatures:—

Temperature.	Resistance. Ohms.
+99·1° C.	7·337
+75·3	6·859
+51·4	6·388
+25·7	5·857
+0·7	5·338
−78·2	3·687
−182·6	1·398
−193·9	1·136
−214·0	0·690

The zero of the thermometer in platinum degrees was −263·27°. Mr. J. D. Hamilton Dickson, M.A., Fellow of Peterhouse, who contributed a paper to the 'Phil. Mag.' for June, 1898, on "The Reduction