explained by the fact that when the monkeys received a banana or an apple every day, they would be less likely to eat as much of the meat as they would otherwise do, and would thus daily receive a smaller dose of ptomaine.

In spite of this fact, in no less than four cases out of the five did we get bloody motions, and in two of the cases spongy gums. In these cases tainted meat alone seems to have produced scorbutic symptoms, although the animals in this group took longer to develop the symptoms, and seemed not to suffer in such a severe form.

Looking at the results of our experiments on monkeys, as a whole, we venture to think that they afford important confirmation of the conclusion derived from Arctic experience, as referred to in the early part of this paper, that the absence or presence of fresh vegetables or lime juice is not alone sufficient for the prevention or the cure of scurvy, and that we must regard the condition of the food in general, and especially the state of preservation of the meat, as the essential factor in the etiology of the disease.

We have to express our thanks to Dr. Francis Goodbody for his untiring assistance in the numerous observations that had to be made during the eighteen months the research was continued.

"The Theory of the Double Gamma Function." By E. W. Barnes, B.A., Fellow of Trinity College, Cambridge. Communicated by Professor A. R. Forsyth, Sc.D., F.R.S. Received February 26,—Read March 15, 1900.

(Abstract.)

The memoir deals with a function which is substantially one-quarter of the σ function of Weierstrass, just as the ordinary (simple) gamma function is substantially one-half of the infinite sine product. The analogy between the two functions determines the nomenclature.

In any development of the simple gamma function from the function-theory point of view, it is necessary to use Euler's theorem

$$\underset{n=\infty}{\mathrm{L}t}\left[1+\tfrac{1}{2}+\ldots\ldots+\frac{1}{n}-\log n\right]=\gamma$$

to obtain the product $\Gamma_1(z/\omega)$ given by

$$\Gamma_1^{-1}(z/\omega) = \omega^{-\frac{z}{\omega}} e^{\frac{\gamma z}{\omega}} \cdot z \cdot \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n\omega}\right)e^{-\frac{z}{n\omega}}\right]$$

as a solution of the difference equation $f(z+\omega) = zf(z)$.

Similarly for the elementary theory of the double gamma function

investigated in Part II of the memoir, two forms, $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$, are considered which arise substantially as finite terms in the approximations for

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} \quad \text{and} \quad \sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{\Omega},$$

where $\Omega = m_1\omega_1 + m_2\omega_2$, when n is large. These approximations are shown to involve logarithms of ω_1, ω_2, and $(\omega_1 + \omega_2)$; and the relative distribution of the points in the Argand diagram representing these quantities causes the introduction of two numbers m and m' of fundamental importance in the theory. The double gamma function $\Gamma_2(z/\omega_1, \omega_2)$ given by

$$\Gamma_2^{-1}(z)\,\omega_1, \omega_2) = e^{\frac{z^2}{2}\gamma_{21}(\omega_1, \omega_2) + z\gamma_{21}(\omega_1, \omega_2)} \, . \, z \, . \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right]$$

is shown to satisfy the two difference relations

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z/\omega_2)}{\rho_1(\omega_2)} e^{-2m\pi i S_1'(z/\omega_2)},$$

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z/\omega_1)}{\rho_1(\omega_1)} e^{-2m'\pi i S_1'(z/\omega_1)}.$$

The functions $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$ are called the first and second double gamma modular forms respectively.

The double gamma function can be expressed in two ways as an infinite product of simple gamma functions; it can be connected with an unsymmetrical function $G(z/\tau)$ first considered by Alexeiewsky; and in terms of it, Weierstrass' elliptic functions can be expressed. By means of the values of the numbers m and m' the well-known relation

$$\eta_1\omega_2 - \eta_2\omega_1 = \pm \tfrac{1}{2}\pi i$$

can be obtained, as well as the fundamental formulæ of the σ function.

Fundamental in the theory of double gamma functions is the double Riemann ζ function, $\zeta_2(s, a/\omega_1, \omega_2)$, which is considered in Part III, and is the simplest solution of the difference equation

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f(a + \omega_2) + f(a) = \frac{1}{a^s},$$

where s, a, ω, ω_1, and ω_2 have any complex values such that ω_2/ω_1 is not real and negative, and a^{-s} has its principal value with respect to the axis of $-(\omega_1 + \omega_2)$. This function is expressible as a contour-integral by means of the relation

$$\zeta_2(s, a) = \frac{i\Gamma(1-s)}{2\pi} e^{2\mathrm{M}s\pi i} \int \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

and the determination of the axis of the contour and the number M depends upon the distribution of the points ω_1, ω_2, -1, and $(\omega_1 + \omega_2)$.

By means of this function we obtain asymptotic approximations for summations of the type

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}$$

when n is a very large number and s has any complex value. We can also obtain an asymptotic approximation for the product

$$\prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a+m_1\omega_1+m_2\omega_2).$$

Since Stirling's theorem gives the asymptotic evaluation of $n\,!$, we obtain, on putting $a = 0$, an extension of Stirling's theorem to two parometers. We find, as the absolute term, the double Stirling function $\rho_2(\omega_1,\omega_2)$, which is the analogue of the simple Stirling form $\rho_1(\omega) = \sqrt{2\pi/\omega}$. All the double asymptotic expansions involve as their coefficients double Bernoullian functions and numbers. The double Bernoullian function ${}_2\mathrm{S}_n(a/\omega_1, \omega_2)$ is an algebraic polynomial which satisfies the two difference equations

$$f(a+\omega_1) - f(a) = \mathrm{S}_n(a/\omega_2) + \frac{\mathrm{S}'_{n+1}(0/\omega_2)}{n+1},$$

$$f(a+\omega_2) - f(a) = \mathrm{S}_n(a/\omega_1) + \frac{\mathrm{S}'_{n+1}(0/\omega_1)}{n+1},$$

and possesses properties exactly analogous to the corresponding simple forms. The theory of this function forms Part I of the memoir. From the contour-integral expression for the double Riemann ζ function it is possible to obtain similar expressions for the logarithm of the double gamma function and its derivatives, for the first and second double gamma modular forms, and for the logarithm of the double Stirling function. Under certain restrictions these contour-integrals can be transformed into line-integrals.

The double gamma function admits of transformation and multiplication theories developed in Part IV. By means of the latter theory we may express the double Stirling form as a product of double gamma functions of arguments $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2}(\omega_1+\omega_2)$ respectively. There is also a transformation theory for the double gamma modular forms and the double Stirling function.

The extension of Raabe's formula for the simple gamma function leads to certain "integral formulæ." The integral

$$\int_0^z \log \Gamma_1(z)\, dz$$

can be expressed in terms of double gamma functions of equal parameters, and the two integrals

$$\int_0^{\omega_1} \log \Gamma_2(z)\, dz, \qquad \int_0^{\omega_2} \log \Gamma_2(z)\, dz$$

can be substantially expressed in terms of the double Stirling function of ω_1 and ω_2.

It is shown in Part V that it is possible to obtain an asymptotic approximation when $|z|$ is very large, for

$$\log \Gamma_2(z+a),$$

which is valid over that part of the region at infinity in which there are no zeros of $\Gamma_2^{-1}(z)$. The coefficients in this expansion are double Bernoullian functions of a.

Finally it is proved that the double gamma function does not satisfy any differential equation whose coefficients are not substantially that function or its derivatives.

There exist n-ple gamma functions whose properties can be obtained by an easy generalisation of the previous theory.

March 22, 1900.

The LORD LISTER, F.R.C.S., D.C.L., President, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

The Croonian Lecture—"On Immunity, with Special Reference to Cell Life"—was delivered by Professor PAUL EHRLICH, of Frankfort-on-the-Main.

March 29, 1900.

The LORD LISTER, F.R.C.S., D.C.L., President, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

The following Papers were read:—