May 2, 1901.

Sir WILLIAM HUGGINS, K.C.B., D.C.L., President, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

In pursuance of the Statutes, the names of the Candidates recommended for election into the Society were read, as follows:—

Alcock, Professor Alfred William, M.B.
Dyson, Frank Watson, M.A.
Evans, Arthur John, M.A.
Gregory, Professor John Walter, D.Sc.
Jackson, Henry Bradwardine, Captain, R.N.
Macdonald, Hector Munro, M.A.
Mansergh, James, M.Inst.C.E.
Martin, Prof. Charles James, M.B.
Ross, Ronald, Major (I.M.S., retired).
Schlich, Professor William, C.I.E.
Smithells, Professor Arthur, B.Sc.
Thomas, Michael R. Oldfield, F.Z.S.
Watson, William, B.Sc.
Whetham, William C. Dampier, M.A.
Woodward, Arthur Smith, F.G.S.

The following Papers were read:—

I. "On the Variation in Gradation of a Developed Photographic Image when impressed by Monochromatic Light of different Wave-lengths." By Sir W. DE W. ABNEY, K.C.B., F.R.S.

II. "Ellipsoidal Harmonic Analysis." By G. H. DARWIN, F.R.S.

III. "On the Small Vertical Movements of a Stone laid on the Surface of the Ground." By HORACE DARWIN. Communicated by CLEMENT REID, F.R.S.

"Ellipsoidal Harmonic Analysis." By G. H. DARWIN, F.R.S., Plumian Professor and Fellow of Trinity College in the University of Cambridge. Received March 23,—Read May 2, 1901.

(Abstract.)

Lamé's functions have been used in many investigations, but the form in which they have been presented has always been such as to render numerical calculation so difficult as to be practically impossible. The object of this paper is to remove the imperfection in question by

giving to the functions such forms as shall render numerical results accessible.

Throughout the work I have enjoyed the immense advantage of frequent discussions with Mr. E. W. Hobson, and I have to thank him not only for many valuable suggestions but also for assistance in obtaining various specific results.

My object in attacking this problem was the hope of being thereby enabled to obtain exact numerical results as to M. Poincaré's pear-shaped figure of equilibrium of rotating liquid. But it soon became clear that partial investigation with one particular object in view was impracticable, and I was led on to cover the whole field, leaving the consideration of the particular problem to some future occasion.

The usual symmetrical forms of the three functions whose product is a solid ellipsoidal harmonic are such as to render purely analytical investigations both elegant and convenient. But it seemed that facility for computation might be gained by the surrender of symmetry, and this idea is followed out in the paper.

The success attained in the use of spheroidal analysis suggested that it should be taken as the point of departure for the treatment of ellipsoids with three unequal axes. In spheroidal harmonics we start with a fundamental prolate ellipsoid of revolution, with imaginary semi-axes $k\sqrt{-1}$, $k\sqrt{-1}$, 0. The position of a point is then defined by three co-ordinates; the first of these, ν, is such that its reciprocal is the eccentricity of a meridional section of an ellipsoid confocal with the fundamental ellipsoid and passing through the point. Since that eccentricity diminishes as we recede from the origin, ν plays the part of a reciprocal to the radius vector. The second co-ordinate, μ, is the cosine of the auxiliary angle in the meridional ellipse measured from the axis of symmetry. It therefore plays the part of sine of latitude. The third co-ordinate is simply the longitude ϕ. The three co-ordinates may then be described as the radial, latitudinal, and longitudinal co-ordinates. The parameter k defines the absolute scale on which the figure is drawn.

It is equally possible to start with a fundamental oblate ellipsoid with real semi-axes k, k, 0. We should then take the first co-ordinate, ζ, as such that $\zeta^2 = -\nu^2$. All that follows would then be equally applicable; but in order not to complicate the statement by continual reference to alternate forms, the first form is taken as a standard.

In the paper a closely parallel notation is adopted for the ellipsoid of three unequal axes. The squares of the semi-axes of the fundamental ellipsoid are taken to be $-k^2\frac{1+\beta}{1-\beta}$, $-k^2$, 0, and the three co-ordinates are still ν, μ, ϕ. As before, we might equally well start with a fundamental ellipsoid whose squares of semi-axes are $k^2\frac{1+\beta}{1-\beta}$, k^2, 0, and replace ν^2 by ζ^2 where $\zeta^2 = -\nu^2$. All possible

ellipsoids are comprised in either of these types by making β vary from zero to infinity. But it is shown that, by a proper choice of type, all possible ellipsoids are comprised in a range of β from zero to one-third. When β is zero we have the spheroids for which harmonic analysis already exists, and when β is equal to one-third the ellipsoid is such that the mean axis is the square root of mean square of the extreme axes. We may then regard β as essentially not greater than one-third, and may conveniently make developments in powers of β.

In spheroidal analysis, for space internal to an ellipsoid ν_0, two of the three functions are the same P-functions that occurs in spherical analysis; one P being a function of ν, the other of μ. The third function is a cosine or sine of a multiple of the longitude ϕ. For external space the P-function of ν is replaced by a Q-function, being a solution of the differential equation of the second kind.

The like is true in ellipsoidal analysis, and we have P- and Q-functions of ν for internal and external space, a P-function of μ, and a cosine- or sine-function of ϕ. For the moment we will only consider the P-functions, and will consider the Q-functions later.

There are eight cases which are determined by the evenness or oddness of the degree i and of the order s of the harmonic, and by the alternative of whether they correspond with a cosine- or sine-function of ϕ. These eight types are indicated by the initials E, O, C, or S; for example, EOS means the type in which i is even, s is odd, and that there is association with a sine-function.

It appears that the new P-functions have two forms. The first form, written $\mathfrak{P}$, is found to be expressible in a finite series in terms of $P_i^{s\pm 2k}$, when the P's are ordinary functions of spherical analysis. The terms in this series are arranged in powers of β, so that the coefficient of $P_i^{s\pm 2k}$ has β^k as part of its coefficient. The second form, written P_i^s, is such that $\sqrt{\dfrac{\nu^2-1}{\nu^2-\frac{1+\beta}{1-\beta}}}\,\mathsf{P}_i(\nu)$ or $\sqrt{\dfrac{1-\mu^2}{\frac{1+\beta}{1-\beta}-\mu^2}}\,\mathsf{P}_i^s(\mu)$ is expressible by a series of the same form as that for $\mathfrak{P}_i^s$. Amongst the eight types four involve $\mathfrak{P}$-functions and four P-functions; and if for given s a $\mathfrak{P}_i^s$-function is associated with a cosine-function, the corresponding P_i is associated with a sine-function, and *vice versâ*.

Lastly, a $\mathfrak{P}$-function of ν is always associated with a $\mathfrak{P}$-function of μ; and the like is true of the P's.

Again, the cosine- and sine-functions have two forms. In the first form s and i are either both odd or both even, and the function written $\mathfrak{C}_i^s$ or $\mathfrak{S}_i^s$ is expressed by a series of terms consisting of a coefficient multiplied by β^k cos or sin $(s \pm 2k)\phi$. In the second form, s and i differ as to evenness and oddness, and the function written C_i^s or S_i is expressed by a similar series multiplied by $(1-\beta\cos 2\phi)^{\frac{1}{2}}$.

The combination of the two forms of P-function with the four forms of cosine- and sine-function gives the eight types of harmonic.

Corresponding to the two forms of P-function there are two forms of Q-function, such that $\mathfrak{Q}_i^s$ and $\mathbf{Q}_i^s \sqrt{\dfrac{\nu^2 - 1}{\nu^2 - \frac{1+\beta}{1-\beta}}}$ are expressible in a series of ordinary Q-functions; but whereas the series for $\mathfrak{P}_i^s$ and $\mathbf{P}_i^s$ are terminable, because P_i^s vanishes when s is greater than i, this is not the case with the Q-functions.

In spherical and spheroidal analysis the differential equation satisfied by P_i^s involves the integer s, whereby the order is specified. So here also the differential equations, satisfied by $\mathfrak{P}_i^s$ or $\mathbf{P}_i^s$ and by $\mathfrak{C}_i^s$, $\mathfrak{S}_i^s$, $\mathbf{C}_i^s$, or $\mathbf{S}_i^s$, involve a constant; but it is no longer an integer. It seemed convenient to assume $s^2 - \beta\sigma$ as the form for this constant, where s is the known integer specifying the order of harmonic, and σ remains to be determined from the differential equations.

When the assumed forms for the P-function and for the cosine- and sine-functions are substituted in the differential equations, it is found that, in order to satisfy the equations, $\beta\sigma$ must be equal to the difference between two finite continued fractions, each of which involves $\beta\sigma$. We thus have an equation for $\beta\sigma$, and the required root is that which vanishes when β vanishes.

For the harmonics of degrees 0, 1, 2, 3 and for all orders σ may be found rigorously in algebraic form, but for higher degrees the equation can only be solved approximately, unless β should have a definite numerical value.

When $\beta\sigma$ has been determined either rigorously or approximately, the successive coefficients of the series are determinable in such a way that the ratio of each coefficient to the preceding one is expressed by a continued fraction, which is in fact portion of one of the two fractions involved in the equation for $\beta\sigma$.

Throughout the rest of the paper the greater part of the work is carried out with approximate forms, and, although it would be easy to attain to greater accuracy, it seemed sufficient in the first instance to limit the development to β^2. With this limitation the coefficients of the series assume simple forms, and we thus have definite, if approximate, expressions for all the functions which can occur in ellipsoidal analysis.

In rigorous expressions $\mathfrak{P}_i^s$ and $\mathbf{P}_i^s$ are essentially different from one another, but in approximate forms, when s is greater than a certain integer dependent on the degree of approximation, the two are the same thing in different shapes, except as to a constant factor.

The factor whereby $\mathbf{P}_i^s$ is convertible into $\mathfrak{P}_i^s$, and $\mathbf{C}_i^s$ or $\mathbf{S}_i^s$ into $\mathfrak{C}_i^s$ or $\mathfrak{S}_i^s$ are therefore determined up to squares of β. With the degree of approximation adopted there is no factor for converting the P's when $s = 3, 2, 1$. Similarly, down to $s = 3$ inclusive, the same factor serves for converting $\mathbf{C}_i^s$ into $\mathfrak{C}_i^s$ and $\mathbf{S}_i^s$ into $\mathfrak{S}_i^s$. But for $s = 2, 1, 0$ one form is needed for changing $\mathbf{C}$ into $\mathfrak{C}$, and another

for changing S into $\mathfrak{S}$. It may be well to note that there is no sine-function when s is zero.

The use of these factors does much to facilitate the laborious reductions involved in the whole investigation.

It is well known that the Q-functions are expressible in terms of the P-functions by means of a definite integral. Hence $\mathfrak{Q}_i^s$ and $\mathbf{Q}_i^s$ must have a second form, which can only differ from the other by a constant factor. The factor in question is determined in the paper.

It is easy to form a function continuous at the surface ν_0 which shall be a solid harmonic both for external and for internal space. Poisson's equation then gives the surface density of which this continuous function is the potential, and it is found to be a surface harmonic of μ, ϕ multiplied by the perpendicular on to the tangent plane.

This result may obviously be employed in determining the potential of an harmonic deformation of a solid ellipsoid.

The potential of the solid ellipsoid itself may be found by the consideration that it is externally equal to that of a focaloid shell of the same mass. It appears that in order to express the equivalent surface density in surface harmonics it is only necessary to express the reciprocal of the square of the perpendicular on to the tangent plane in that form. This result is attained by expressing x^2, y^2, z^2 in surface harmonics. When this is done an application of the preceding theorem enables us to write down the external potential of the solid ellipsoid at once.

Since x^2, y^2, z^2 have been found in surface harmonics, we can also write down a rotation potential about any one of the three axes in the same form.

The internal potential of a solid ellipsoid does not lend itself well to elliptic co-ordinates, but expressions for it are given.

If it be desired to express any arbitrary function of μ, ϕ in surface harmonics, it is necessary to know the integrals, over the surface of the ellipsoid, of the squares of the several surface harmonics, each multiplied by the perpendicular on to the tangent plane. The rest of the paper is devoted to the evaluation of these integrals. No attempt is made to carry the developments beyond β^2, although the methods employed would render it possible to do so.

The necessary analysis is difficult, but the results for all orders and degrees are finally obtained.