4. Higher mammals depend principally upon variations in heat-loss, in which rapid respiration plays an important part.

5. Variation in production of heat is the ancestral method of homothermic adjustment. During the evolution of the warm-blooded animal it has, through developing a mechanism by means of which it can vary production in accordance with heat lost, overcome one disadvantage of cold-blooded animals, viz., that activity is dependent on external temperature. It has thereby increased its range in the direction of low temperatures. Later, by developing a mechanism controlling loss of heat, it has increased its range in the direction of high temperatures, and also rendered body temperature largely independent of activity; these advantages have been gained by a greater expenditure of energy.

"On the Elastic Equilibrium of Circular Cylinders under certain Practical Systems of Load." By L. N. G. Filon, M.A., B.Sc., Research Student of King's College, Cambridge; Fellow of University College, London; 1851 Exhibition Science Research Scholar. Communicated by Professor Ewing, F.R.S. Received May 20,—Read June 6, 1901.

(Abstract.)

The paper investigates solutions of the equations of elasticity in cases of circular symmetry, and it applies them to discuss the elastic equilibrium of the circular cylinder under systems of surface loading which do not lead to the simple distributions of stress usually assumed in practice.

The analytical method employed has been to solve the equations of elasticity in cylindrical co-ordinates, obtaining solutions in the typical form $\begin{matrix}\cos\\ \sin\end{matrix}\left\{kz\right\} \times$ (function of r), r being the distance from the axis and z the distance measured along the axis.

More general solutions, not necessarily symmetrical about the axis, have been given by Professor L. Pochhammer* and by Mr. C. Chree.† Professor Pochhammer has used his results to deduce approximate solutions for the bending of beams. Neither Mr. Chree nor Professor Pochhammer has, so far as I am aware, worked out his solutions in detail for such problems as are discussed in the present paper.

I found that solutions in trigonometrical series would be sufficient to satisfy most conditions in the first of the three cases discussed, and all

* 'Crelle's Journal,' vol. 81.

† 'Cambridge Phil. Soc. Trans.,' vol. 14.

conditions in the third. The second case required the introduction of other typical solutions, and the analysis was more intricate.

The three problems investigated are as follows:—

In the first I consider a cylinder under pull, the pull not being applied by a uniform distribution of tension across the plane ends, but by a given distribution of axial shear over two zones or rings, towards the ends of the cylinder.

The second is that of a short cylinder compressed longitudinally between two rough rigid planes, in such a manner that the ends are not allowed to expand.

The third case is that of the torsion of a bar in which the stress is applied, not by cross-radial shears over the flat ends, as the ordinary theory of torsion assumes, but by transverse shears over two zones or rings of the curved surface.

The first problem corresponds to conditions which frequently occur in tensile tests, namely, when the piece is gripped by means of projecting collars, the pull being in this case transmitted from the collar to the body of the cylinder by a system of axial shears.

Analytical solutions are found when this system of axial shears is arbitrarily given, there being given also an arbitrary system of radial pressures. Approximate expressions are deduced when the length of the cylinder is large compared with its diameter. These show that the strains and stresses may be calculated on the assumption that we have, over any cross-section, a uniform tension across the section, a constant radial pressure and an axial shear proportional to the distance from the axis, the last two occurring only over the lengths of the cylinder where such stresses are applied. The effects of local pressure and shear are thus, for a long cylinder, restricted to a small region and, in the free parts of the bar, we have, to this approximation, the state of things assumed by the ordinary theory.

In order, however, to study the effect of such a system of surface stresses, when no approximations are involved, I have worked out numerically a case where there is no radial pressure applied externally, and a uniform axial shear is applied between two zones. The solution gives zero tension across the plane ends; it is not, however, found possible to fulfil completely the condition of no stress, and we have over these limiting planes a self-equilibrating system of radial shears, which, however, will produce little effect at a distance from the ends. The length of the cylinder is taken to be $\pi/2$ times the diameter, this ratio being found to simplify the arithmetic. The two rings of shear extend each over one-sixth of the length and are at equal distances from the mid-section and the two ends.

In this and the other numerical examples, Poisson's ratio has been taken as one-fourth. This is not correct for most materials, but as the object was to find out the differences between the results of the simple

and the modified theories, rather than to calculate the absolute stresses and displacements for any given material, the exact value of Poisson's ratio adopted was comparatively unimportant.

It is then found that the stress is greatest at the points where the shear is discontinuous, *i.e.*, at the ends of the collar in a practical case. At these points it is theoretically infinite. This result is true whatever the dimensions of the cylinder. For materials like cast iron or hard steel, which are brittle, such points would therefore be those of greatest danger; but in such a case as that of wrought iron or mild steel, for instance, the stress will be relieved by plastic flow.

The tensile stress varies considerably over the cross-section, and the distortion of the latter is large. Towards the middle of the bar, the axial displacement at the surface is, roughly, twice what it is at the centre.

In tensile experiments the elongation is usually measured by the relative displacement of two points on the outer skin of the cylinder, as recorded by an extensometer. When the test-piece is seized in this way, the surface stretches more than the interior, and consequently a negative correction should be applied to the readings of the extensometer. In the somewhat extreme case considered, this correction may amount to as much as 30 per cent.

The lateral contraction is very much smaller than the theory of uniform tension indicates, being in fact never so great as 60 per cent. of the amount calculated on that hypothesis. For points inside the material the discrepancy is still greater. These variations appear due to the fact that there are considerable radial and cross-radial tensions inside the material, these tensions being often equal to about one-fifth of the mean tension Q, which would give the same total pull.

Tables are given in the paper showing the values of the radial and axial displacements u and w, and of the four stresses $\widehat{rr}$, $\widehat{zz}$, $\widehat{rz}$, $\widehat{\phi\phi}$ (in the notation of Todhunter and Pearson's 'History of Elasticity,' $\widehat{st}$ being the stress, parallel to s, across a face perpendicular to t) for points in the cylinder at distances from the axis $= 0$, $\cdot 2a$, $\cdot 4a$, $\cdot 6a$, a; a being the radius of the cylinder; and for intervals of length parallel to the axis equal to tenths of the half-length. These tables are illustrated by curves and diagrams.

The second problem is of considerable importance, as it illustrates the crushing of blocks of cement or stone, when they are compressed between iron planes, or between sheets of mill-board, so that their ends are constrained not to expand.

The analytical solution is made up, partly of a finite number of terms which are algebraic and rational in r and z, and partly of infinite series involving sines and cosines containing z. By suitably combining these two types of terms all the conditions can be satisfied.

The numerical example taken was one in which the length is nearly

equal to the diameter—the exact ratio, $\pi/3$, being chosen so as to simplify the arithmetic as far as possible.

As in the preceding example, tables of the stresses are given for a large number of points in the cylinder. From these the principal stresses and the principal stretch were calculated; and again from these, by interpolation, curves were drawn showing the loci of points in the cylinder where the greatest stress, the greatest stretch, or the greatest stress-difference had the same value.

The curves show that, whatever theory of yielding is adopted, namely, the greatest-stress theory of Navier and Lamé, or the greatest strain-theory of St. Venant, or the greatest stress difference (or greatest shear) theory which has more recently been put forward, failure of elasticity will begin to take place round the perimeter of the plane ends.

Thus, in the case of the stress, consider the regions where the stress is greater than a certain value S. When S is nearly equal to the greatest stress these regions are thin annuli round the ends. As S diminishes the regions become made up, partly of such annuli (of increasing thickness), partly of a closed region round the centre of the cylinder. When S reaches a certain critical value, S_0, these two regions join on to one another. The regions where the stress is less than S_0 consist of caps at the two ends and of cylindrical shells, forming the "skin" of the cylinder.

The regions of least stress consist only of caps or buttons of material at the two ends.

The variations of the principal stretch and of the principal stress-difference can be described in the same general terms.

For materials like stone and cement, which have no very definite yield point, the elastic distribution will give at least an indication of the state of stress almost up to the point of rupture, and if it be assumed that the latter takes place over the regions of greatest stress, or greatest strain, or greatest shear, according to the particular theory we adopt, the results above show that the fracture will start from the perimeter of the ends, and that caps or buttons, which may have an approximately conical shape, will probably be cut off at the ends.

The fact that yielding first occurs at the perimeter, when the stress exceeds 1/1·686 of the limiting stress for uniform pressure, leads to the conclusion that the strength of a cylinder under this system of stress is considerably less than the strength of a cylinder uniformly compressed. This result apparently contradicts the fact that the strength of stone and cement, when tested between lead plates, which allow of expansion, is very much less than when tested between mill-board which does not allow of expansion, a fact which has led Professor Perry to state that the true strength of such materials is about half their published strength. ('Applied Mechanics,' p. 345.)

The contradiction, however, seems to be explained by a remark of

Unwin's ('Testing of Materials of Construction,' p. 419), which is corroborated by Professor Ewing, to the effect that lead, which is a plastic material and flows easily, not only does not hinder expansion of the ends of the block, but *forces* it.

It is shown in the paper that, under such conditions, whenever the forced expansion exceeds the natural lateral expansion of the stone or cement, which it practically always does, then the points of failure, instead of being at the perimeter of the ends, are at the centre, and the limiting stress, under these circumstances, may be much less than that obtained for non-expanding ends. Further, this limiting stress depends upon the amount of flow of the lead and has no fixed value—a conclusion confirmed by the experimental results of Unwin. The mill-board test, on the other hand, should give consistent results, although it really introduces too large a factor of safety. The change in the form of the fracture, noticed by Unwin, is also accounted for by theory.

The values of the apparent Young's modulus and of the apparent Poisson's ratio are investigated. Young's modulus is shown to vary between its true value, when the cylinder is long, and the value of the ratio of stress to axial contraction, when lateral expansion is prevented by a suitable pressure, this last corresponding to the case when the cylinder is made very short.

In the given example, Poisson's ratio is apparently 0·269, the actual value assumed being 0·25. It should diminish down to zero as the cylinder becomes indefinitely short.

The third problem corresponds to the case of a cylinder whose ends are surrounded by a collar so that the applied torsion couple is transmitted to the inner core by means of transverse shear.

A general solution is first found for a given arbitrary system of transverse shear. Approximate expressions are given when the length of the cylinder is large compared with its diameter. These show that, to the first approximation, the cross-sections remain undistorted, radii originally straight remaining so. The shear across the section, at any point of it, is connected with the total torsion moment at that section by the same relation as in the ordinary theory of torsion. A transverse shear $\widehat{r\phi}$ varying as the square of the distance from the axis exists over the lengths of the cylinder subjected to external stress.

As a numerical example a cylinder is considered, whose length is $\pi/2$ times its diameter, and which is subjected, over lengths at the ends, each equal to one-fourth of the whole length, to a uniform transverse shear. Using the exact expressions found, the stresses and transverse displacement are calculated for various points, and these are compared with the values calculated from the approximate expressions when the cylinder is long.

It is found that the agreement is, on the whole, tolerably good, whence it is inferred that in torsion, the effect of local action dies out more rapidly than in tension or compression. The only case of obvious divergence is with regard to the shear $\widehat{r\phi}$. This shear persists inside, even at sections where no stress of this kind is applied to the outside of the cylinder, but it continually diminishes as we recede from the ends.

In the exact solution, the cross-sections do not remain undistorted, the transverse displacement increasing more rapidly than the radius. The distortion is small at sections where there is no external applied stress, but is very obvious near the ends.

Further, when the applied transverse shear varies discontinuously, as in this case, the other stress becomes infinite at the points of discontinuity. This suggests why it is that abrupt changes in the section of such a cylinder are dangerous. The projecting parts acting upon the inner core will introduce a sudden change in the transverse shear. It has been noticed that propeller shafts usually break at such points.

"The Measurement of Ionic Velocities in Aqueous Solution, and the Existence of Complex Ions." By B. D. STEELE, B.Sc., 1851 Exhibition Scholar (Melbourne). Communicated by Professor RAMSAY, F.R.S. Received May 10,—Read June 6, 1901.

(Abstract.)

The method of measuring ionic volocities described by Masson has been extended in such a manner that, by the present method, the use of gelatin solution and of coloured indicators is not necessary.

An aqueous solution of the salt to be measured is enclosed between two partitions of gelatin which contain the indicator ions in solution, the apparatus being always so arranged that the heavier solution lies underneath the lighter. On the passage of the current the ions of the measured solution move away from the jelly, followed at either end by the indicator ions; the boundary is quite visible in consequence of the difference in refractive index of the two solutions. The velocity of movement of the margins is measured by means of a cathetometer, and the ratio of the margin velocities gives at once the ratio of the ionic velocities.

It is found that, for the production and maintenance of a good refractive margin, a certain definite range of potential fall is required for any given pair of solutions, and this range differs very much for different boundaries—for example, the margin potassium acetate