nion function, and this function is employed in the investigation of the properties of a four-system of linear transformations, of the general quadratic transformation, and of the non-linear one-to-one correspondence of points in space. The method of quaternion arrays* is applied to the discussion of n-systems of linear transformations, and of the critical assemblages of points, lines and planes connected with each system of transformations. Finally, in the concluding section it is explained how the method of the paper may be applied to hyper-space, or to the discussion of functions of any number of variables; and in many cases the formulæ obtained in the course of the paper with special reference to three dimensions require no modification to fit them for the general case of n variables.

"The Stability of the Pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid." By G. H. Darwin, F.R.S

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(Abstract.)

At the end of a previous paper† it was stated that the stability of the pear-shaped figure could not be definitely proved without further approximation. After some correspondence with M. Poincaré during the course of my work on that paper, I attempted to carry out the second approximation, but found myself foiled at a certain stage of the work. Meanwhile he had turned his attention to the subject, and he has‡ shown how the difficulty which stopped me may be overcome. He has not, however, pursued the arduous task of converting his results into numbers, so that he leaves the question of stability unanswered.

M. Poincaré was so kind as to allow me to detain his manuscript on its way to the Royal Society for a few days, and being thus assisted I was able to resume my attempt under favourable conditions, and this paper is the result. The substance of the analysis of this paper is, of course, essentially the same as his, but the two present but little superficial resemblance. It is perhaps well that the two investigations of so complicated a subject should be nearly independent of one another.

If a mass of liquid be rotating like a rigid body with uniform angular velocity, the determination of the figure of equilibrium may be

* 'Trans. R. Irish Acad., vol. 32, pp. 17—30.

† 'Phil. Trans.,' A, vol. 198, pp. 301—331.

‡ 'Phil. Trans.,' A, vol. 198, pp. 333—373.

treated as a statical problem, if the mass be subjected to a rotation potential. The energy lost in the concentration of such a system from a condition of infinite dispersion consists of two parts. The first of these, say W, is the lost energy of the system at rest; the second is equal to the kinetic energy, say T, of the system in motion. The whole lost energy, say E, is equal to $W+T$, and the condition for a figure of equilibrium is that E shall be stationary for all variations, subject to constant angular velocity.

It might appear at first sight that the condition for secular stability is that E shall be a maximum. But M. Poincaré has shown that this condition is insufficient, and that it is necessary for stability that the whole energy, say U, which is equal to $-W+T$, shall be a minimum for all variations, subject to the condition of constancy of angular momentum.

He has, however, adduced another consideration, which enables us to determine the stability from the variations of E, without a direct consideration of the function U. He has shown, in fact, that if for given angular momentum slightly less than that of the critical Jacobian ellipsoid, from which the pear-shaped figures bifurcate, there is only one possible figure, namely, the Jacobian; and if for slightly greater angular momentum there are two figures, namely, the Jacobian and the pear,* then exchange of stability between the two series must occur at the bifurcation. If, on the other hand, the smaller momentum corresponds with the two figures and the larger with only one, one of the two (namely, the Jacobian) must be stable, and the other (namely, the pear) unstable.

The question is then completely answered by the value of the momentum of the pear; if it is greater than that of the critical Jacobian, the pear is stable, and if less, unstable. It suffices then to determine the pear from the variations of E with constant angular velocity, and afterwards to evaluate the angular momentum.

In the first approximation the pear-shaped figure is represented by the third zonal harmonic inequality with reference to the longest axis of the critical Jacobian ellipsoid. In proceeding to the higher approximation I suppose that its amplitude is measured by a parameter e, which is to be regarded as a quantity of the first order. We must now also suppose the ellipsoid to be deformed by every other harmonic, but with amplitudes of order e^2. In the first approximation W was proportional to e^2, but it now becomes necessary to go as far as the order e^4. A change in the sign of e means that the figure is rotated in azimuth through 180°. As this rotation cannot affect the energy, the odd powers of e must be absent from the expression for W. We have further to find the moment of inertia, as far as the terms

* For the sake of simplicity, I speak of one pear instead of two in azimuths differing by 180°.

of order e^2, and thence to find the kinetic energy T. The function E is then equal to W + T.

In order to attain the requisite degree of accuracy it is convenient to regard the pear as being built up in an artificial manner.

I construct an ellipsoid similar to and concentric with the critical Jacobian, and therefore itself possessing the same character. The size of the new ellipsoid, which I call J, is undefined; and is subject only to the condition that it shall be large enough to enclose the whole pear. The region between J and the pear being called R, I suppose the pear to consist of positive density throughout J and negative density throughout R.

The lost energy of the pear consists of that of J with itself, say $\frac{1}{2}$JJ; of J with R, which is filled with negative density, say $-$JR; and of $-$R with itself, say $\frac{1}{2}$RR. This last contribution (which had baffled me) must be broken into several parts.

If we imagine J to be intersected by a family of orthogonal curves, and if we suppose for the moment that the region R is filled with positive matter, we may further imagine the matter lying inside any orthogonal tube to be transported along the tube, and deposited on the surface of J in the form of a concentration of positive surface density +C.

In the actual system R is filled with negative density, and we may clearly add to this two equal and opposite surface densities +C and $-$C on J. The matter lying in the region R may then be regarded as consisting of negative surface density $-$C, together with a double system, namely negative volume density $-$R, conjoined with equal and opposite surface density +C. This double system, say D, is therefore C $-$ R.

The lost energy $\frac{1}{2}$RR may now be considered as consisting of three parts, first, the energy of $-$C with itself, say $\frac{1}{2}$CC; secondly, that of D with itself, say $\frac{1}{2}$DD; and thirdly of $-$C with D. This third item is obviously equal to $-$CC + CR, and therefore $\frac{1}{2}$RR is equal to $-\frac{1}{2}$CC + CR + $\frac{1}{2}$DD. Thus W, the gravitational lost energy of the pear, may be written symbolically—

$$\tfrac{1}{2}\mathrm{JJ} - \mathrm{JR} + \mathrm{CR} - \tfrac{1}{2}\mathrm{CC} + \tfrac{1}{2}\mathrm{DD}.$$

In this discussion no attention has as yet been paid to the rotation, but fortunately it happens that the introduction of this consideration actually simplifies the problem, for if we suppose $\frac{1}{2}$JJ and JR to mean the lost energies of J with itself and with R on the supposition that the mass is rotating with the angular velocity of the critical Jacobian, the formulæ become much more tractable than would otherwise have been the case.

The inclusion of part of the angular velocity in this part of the function only leaves outstanding the excess of the kinetic energy of

the pear above the kinetic energy which it would have had if it rotated with the angular velocity of the critical Jacobian. If ω denotes the latter angular velocity, and $(\omega^2+\delta\omega^2)^{\frac{1}{2}}$ the actual angular velocity of the pear; if A_j, A_r denote the moments of inertia of J, and of R considered as filled with positive density, we have

$$E = \tfrac{1}{2}JJ - JR + CR - \tfrac{1}{2}CC + \tfrac{1}{2}DD + \tfrac{1}{2}(A_j - A_r)\,\delta\omega^2.*$$

The co-ordinates of points are determined by reference to the ellipsoid J which envelopes the whole pear. The size of J is indeterminate, and therefore the formulæ must involve an arbitrary constant expressive of the size of J. But the final result for E cannot in any way depend on the size of the ellipsoid which is chosen as the basis for measurement, and therefore the arbitrary constant must ultimately disappear. Hence it is justifiable to treat it as zero from the beginning, and we may use the formula for the internal gravity throughout the investigation.†

Although the constant expressive of the size of J is put equal to zero—which means that the pear is really partly protuberant beyond the ellipsoid—yet there is a considerable amount of mental convenience in continuing to discuss the subject as though the ellipsoid completely enveloped the pear.

When an ellipsoid is deformed by an harmonic inequality, the volume of the deformed body is only equal to that of the ellipsoid, to the first order of small quantities. In the case of the pear, all the inequalities, excepting the third zonal one, are of the second order, and as far as concerns them the volumes of J and of the pear are the same. But it is otherwise as regards the third zonal harmonic term, and the first task is to find the volume of such an inequality as far as e^2. When this is done, we can express the volume of J in terms of that of the pear, which is of course a constant.

By aid of ellipsoidal harmonic analysis we may now express the first four terms of E in terms of the mass of the pear and of certain definite integrals which depend on the shape of the critical Jacobian ellipsoid.

The energy $\frac{1}{2}$DD presents much more difficulty, and it is especially in this that M. Poincaré's insight and skill have been shown. The system D consists of a layer of negative volume density coated on its outer surface with a layer of surface density of equal and opposite mass. His procedure virtually amounts to regarding this system as consisting of an infinite number of magnetic layers, whose energy may be evaluated and summed. The reduction of this part of the energy to calculable forms is not very simple.

* A term depending on the shift of the centre of inertia proves to be negligible.

† Compare with M. Poincaré's treatment of the same point, 'Phil. Trans.,' A, vol. 198, p. 352.

The moment of inertia of the pear presents but little difficulty, since it only involves those harmonic inequalities of J which are expressible by harmonics of the second degree. On multiplying the moment of inertia by $\frac{1}{2}\delta\omega^2$ we obtain the last contribution to the expression for E.

The portion of E independent of $\delta\omega^2$ cannot involve e^2, since the vanishing of the coefficient of that term is the condition whence the critical Jacobian ellipsoid was determined. If f denotes the coefficient of any harmonic inequality other than the third zonal one, this portion of E is found to contain terms in e^4, e^2f, and $(f)^2$. The coefficient of $\delta\omega^2$ consists of a constant term and terms in e^2, f_2, $f_2{}^2$, where these f's denote the coefficients of the second zonal and sectorial harmonics. If f refers to any harmonic of odd degree, the coefficient of the corresponding term in e^2f vanishes. If, then, we make E stationary for variations of the coefficient of any odd harmonic, that coefficient is seen to vanish. Hence it follows that the expression for the pear cannot involve any odd harmonic other than the third zonal one. Conditions of symmetry also negative the existence of even harmonics of the sine type, and of even harmonics of the cosine type but of odd rank.

On equating to zero the variations of E for all the remaining f's, excepting f_2 and $f_2{}^2$, we at once obtain their values in terms of e^2. Equating to zero the variations for e^2, f_2, $f_2{}^2$, we obtain three equations, which give $\delta\omega^2$, f_2, $f_2{}^2$ as multiples of e^2.

It seems unnecessary to explain here the methods adopted for reducing the analytical results to numbers; it may suffice to say that the task was very laborious.

The harmonic terms included in the computation were those of degree 2, and ranks 0, 2; of degree 4, and ranks 0, 2, 4; and of degree 6, and ranks 0, 2, 4. The sixth sectorial harmonic would certainly have proved negligible.

The expression for $\delta\omega^2$ was found in the form of a fraction, of which the denominator is determinate, and the numerator is the sum of an infinite series. Nine terms of this series were computed, namely, a constant term and the contributions of the eight harmonics above enumerated. The result shows that the square of the angular velocity of the pear is less than that of the Jacobian in about the proportion $1-\frac{1}{8}e^2$ to 1.

On the other hand, the angular momentum is greater in about the proportion of $1+\frac{1}{15}e^2$ to 1. If this last result were based on a rigorous summation of the infinite series, it would absolutely prove the stability of the pear. The inclusion of the uncomputed residue of the series would undoubtedly tend in the direction of reducing the coefficient given above in round numbers as $\frac{1}{15}$, and if it were to reduce it to a negative quantity we should conclude that the pear is unstable after all.

The apparently rapid convergence of the series seemed to render such a reversal of the result almost incredible. In order, however, to feel yet more sure, I made a rough estimate of the contribution of the eighth zonal harmonic, and found that it would only amount to $\frac{1}{175}$th part of that critical total which would just show the pear to be unstable.

Since the convergency of the series is obviously rapid, I regard it as proved, but by something short of absolute algebraic proof, that the pear is stable.

The numbers obtained in the course of the work afford the means of giving a second approximation to the form of the pear, and the result is shown in figures, drawn with the largest value of e, which seemed consistent with a fair degree of approximation.

I originally called the figure "pear-shaped" because M. Poincaré's conjectural sketch in the 'Acta Mathematica' was very like a pear. In the first approximation, shown in my former paper, the resemblance to a pear was not striking, and it needs some imagination to see the pear-shape in the new figures; but a distinctive name is so convenient that we may as well continue to call it by that name.

The effects of the new terms are almost entirely concentrated at the two ends. They tend to augment the protuberance of the stalk end, and to diminish the depression at the blunt end so much as nearly to fill it up. Over the greater part of the figure the depressions and protuberances are less conspicuous than they were.

I think it is hardly too much to say that in a well-developed "pear" the Jacobian ellipsoid has nearly regained its primitive figure, but that it has shot forth a protuberance at one end. A consideration of the figures and of a conjectural extension of them almost reminds one of some such phenomenon as the protrusion of a filament of protoplasm from a mass of living matter. Notwithstanding the warning of M. Poincaré as to the danger of applying these results to heterogeneous masses and thence to cosmogony, I cannot restrain myself from joining him in seeing in this almost life-like process a counterpart to at least one form of the birth of double stars, planets, and satellites.