*Appendix IV.*—The following values have been obtained for the vapour pressures of solid neon :—

| Temperature (helium scale). | Pressure (millimetres). |
|---|---|
| 20°·4 | 12·8 |
| 15·65 | 2·4 |

The vapour pressure of the neon did not change after successive portions of it had been allowed to evaporate. This proved that neon is a homogeneous substance.

*Appendix V.*—From consideration of the periodic relationship between the critical and boiling points of the elements of the helium-argon group it appears probable that the critical point of helium lies at about 10°·5 Abs. and the boiling point at 6° Abs.; Dewar fixes* the former of these points at below 9° or 10°, and the latter at about 5°. In a series of experiments helium was compressed into a tube, one end of which was cooled in liquid or solid hydrogen. At temperatures down to that of solid hydrogen evaporating under a pressure of 5 mm. of mercury (probably about 13° Abs.), the pressure on the helium was slowly increased to 60 atmospheres. Under all conditions a change of pressure accompanied a change of volume of the gas, and no evidence that liquefaction had taken place could be obtained.

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"On an Approximate Solution for the Bending of a Beam of Rectangular Cross-section under any System of Load, with Special Reference to Points of Concentrated or Discontinuous Loading." By L. N. G. FILON, B.A. (Cantab.), M.A., B.Sc. (Lond.), King's College, Cambridge, Fellow of University College, London, and 1851 Exhibition Science Research Scholar. Communicated by Dr. C. CHREE, F.R.S. Received June 12,—Read June 19, 1902.

(Abstract.)

The paper investigates the elastic equilibrium of a long bar of rectangular cross-section in those cases where the problem may be treated as one of two dimensions, namely :—

(*a*.) When the strain being in the plane of $xy$, the elastic solid extends indefinitely in the direction of $z$, the applied stresses over the bounding planes $y = \pm b$, $x = \pm a$ being the same for any two sections parallel to the plane of $xy$. We then have a strictly two-dimensional strain.

* *Loc. cit.*, p. 364.

(*b*.) When, on the other hand, the applied stresses still being in the plane of $xy$, the thickness of the bar, in the direction of the axis of $z$, is small compared with the other dimensions of the bar, so that we approximate to the case of a thin plate under thrust in its own plane.

It is shown that, if in case (*b*) we assume (which will be very nearly true, the thinner the lamina) that the normal traction across a face perpendicular to $z$ is zero throughout the thickness, then the equations connecting the *mean* displacements U, V with the *mean* stresses P, Q, S in the plane of the lamina (the mean here being taken with regard to the thickness of the lamina) are of the same form as the equations in case (*a*) connecting the actual displacements $u$, $v$ with the three stresses in the plane of $xy$, provided only that we make an alteration in one of the elastic constants, $u$, $v$, $w$ being displacements parallel to $x$, $y$, $z$ according to the usual notation.

Of course, the lamina being thin, the displacements $u$, $v$ will probably vary little as we go across it, so that the mean values will give us an approximation to the displacements at every point.

In like manner the stresses in the planes parallel to $xy$ will not differ greatly from their mean values P, Q, S.

The equations used in the paper correspond to case (*b*), but all the results are applicable to case (*a*) by merely changing the elastic constant mentioned. The body stress equations are

$$\frac{dP}{dx}+\frac{dS}{dy} = 0, \qquad \frac{dS}{dx}+\frac{dQ}{dy} = 0,$$

where

$$P = \lambda'\left(\frac{dU}{dx}+\frac{dV}{dy}\right)+2\mu\frac{dU}{dx},$$

$$Q = \lambda'\left(\frac{dU}{dx}+\frac{dV}{dy}\right)+2\mu\frac{dV}{dy},$$

$$S = \mu\left(\frac{dU}{dy}+\frac{dV}{dx}\right),$$

where $\lambda' = 2\lambda\mu/(\lambda+2\mu)$ and $\lambda$, $\mu$ are the elastic constants of Lamé.

General solutions of these equations are found in terms of conjugate functions. These solutions are then applied to the case of a rectangular bar bounded by the planes $x = \pm a$, $y = \pm b$.

The surface stresses applied to the faces $y = \pm b$ are supposed given at every point, but over the faces $x = \pm a$ only the statical stress-resultants (total tension, total shear, total bending moment) are supposed given.

This last condition is sufficient, provided $a$ is large compared with $b$. This is assumed in every case; eventually the boundaries $x = \pm a$ are removed to infinity.

The first part of the paper is occupied in establishing the formal

solution for the most general system of applied stress of the above type when $a$ is finite. This is found to lead to infinite series of the form

$$\Sigma\,(a_n + b_n y) \begin{Bmatrix} \cosh \\ \sinh \end{Bmatrix} \frac{n\pi y}{a} \times \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \frac{n\pi x}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$n$ being a positive integer, and $a_n$, $b_n$ arbitrary constants. Together with these infinite series, there enter into the solutions a finite number of terms of the form

$$c_{mn} x^m y^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

These represent solutions for certain cases where the body stress equations can be solved in finite terms, so as to give zero stress over the boundaries $y = \pm b$. For instance, a uniform tension parallel to $x$, a uniform bending moment, and a uniform shear give rise to solutions of this type. They can be superimposed upon the others without affecting the stress distributions over $y = \pm b$, and they are introduced to satisfy the terminal "total" conditions.

In the various cases considered, the length $a$ of the beam is allowed to tend to infinity. The series then degenerate into integrals. The transformation and interpretation of these integrals are dealt with at length. It is shown that they may be expanded in series of the form

$$\Sigma\,(d_n + e_n y)\, r^n \cos n\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$r$, $\phi$ being polar co-ordinates about any point in the beam as origin, $n$ being an integer, and $d_n$, $e_n$ being constants, which are determined.

The form of these series varies with the origin chosen. When the origin is a point where a concentrated load is applied, the series for the stresses start with a negative value of $n$, giving terms which become infinite when $r = 0$.

In this case the corresponding series for the displacements contain terms in $\log r$ and $\phi$, which lead to discontinuities and infinities. These of course could not occur in any actual problem, but in practice the material immediately below a concentrated load would probably become plastic, so that in the immediate neighbourhood of such loads the solution will not apply.

It is found that the terms involving infinities and discontinuities are precisely those to which the solution reduces, when the height $2b$ is made very large. They agree with the solutions given by Boussinesq and Flamant* for two-dimensional strains in an infinite solid bounded by a plane and subjected to load concentrated along a straight line.

The series of terms involving positive powers of $r$ represent therefore the corrections to Boussinesq's expressions, when the finite height of the beam is taken into account.

* 'Comptes Rendus,' vol. 114, pp. 1465—68 and pp. 1510—16.

The various cases, which are separately dealt with, are as follows :—

(1.) When the external stresses upon the top and bottom faces $y = \pm b$ are purely normal and are symmetrical about the mid-section, $x = 0$.

In the first place, making first $a$ large, but not infinite, the various terms of the sine and cosine series (1) may be expanded in terms of $y/a$. Approximate expressions are then obtained for the displacements and stresses in a very long beam, at a distance from the regions near the points where the loads are applied. The validity of such expansions has been discussed in a paper by the author "On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load."*

The results in the present case show that to this approximation the stresses P, S are given in terms of the total bending moment and total shear by the formulæ given by de Saint-Venant for a beam terminally loaded, but otherwise free.

In the case of the displacements, however, it is found that, for a doubly supported beam under a central isolated load, the vertical deflection of the central axis contains a term $-kx$ where $x$ is positive, and $+kx$ where $x$ is negative. Such a term was put in by de Saint-Venant for a built-in beam. Professor Love, starting from different conditions for a built-in end, arrived at the conclusion that the term should be zero.

As a matter of fact the term is found to exist, but the coefficient $k$ is only 0·74 of de Saint-Venant's value, showing that in passing an isolated load the slope of the elastic line varies fairly abruptly, but only to about three-fourths of the extent anticipated by de Saint-Venant.

The variations in the central deflection, as the supports are brought closer and closer together, are also investigated. It is found that the excess of the actual over the Euler-Bernoulli deflection (which excess is sometimes referred to by engineers as the "deflection due to shear") decreases eventually as the span decreases and, for exceedingly small spans, may even become negative.

The series in powers of $r$, deduced from the other expressions when $a$ is made infinite, are used to show the variations of stress in the mid-section and the results are compared with those obtained by Sir G. Stokes† and Boussinesq‡ from an empirical formula. It is shown that, though the empirical formula gives an approximation to the stress in some places, it is by no means to be relied upon.

The case of a beam under two opposite isolated loads, which leads at once to the more interesting problem, of a beam carrying an isolate

* 'Phil. Trans.,' A, vol. 198, pp. 147—233.

† 'Phil. Mag.,' ser. 5, vol. 32, pp. 500—503.

‡ *Loc. cit.*

load and resting upon a smooth rigid plane, is next considered. The distribution of the pressure upon the plane is investigated and a new form of expansion found for it. It is shown that, outside a certain limited area below the load, a *tension* is required to keep the elastic solid in contact with the plane, so that such a solid would be lifted at the sides, by applying pressure at the centre.

(2.) When the stresses across $y = \pm b$ are still normal, but are asymmetrical with regard to $x = 0$.

In particular the behaviour of a beam under two concentrated loads acting in opposite senses upon opposite faces of the beam, their lines of action being on opposite sides of the mid-section, is studied. The manner in which the shear across the middle section varies as these loads are made to approach each other is exhibited by various diagrams. These show how rapidly the effects of the particular distribution of any total terminal load die out as we go away from the end. At a distance of the same order as the height of the beam, they already begin to be negligible.

At a lesser distance than this, however, such effects may become exceedingly important. The case of rivets is instanced, and it is suggested that the results obtained in the paper may give some information which shall be useful in this connection.

(3.) When the stresses across $y = \pm b$ are purely tangential. The special case here treated is that where these stresses reduce to a single concentrated tangential force.

As in practice we cannot approximate to a line distribution of shearing stress, the effect of spreading it out over an area is investigated. It is then found that, though the displacements are everywhere finite and continuous, a discontinuity (though not an infinity) in the surface shear leads to an infinite stress at the point, and is therefore a source of danger to the material.

It is found also that shear depresses those parts of the solid *towards* which it acts. Both these results agree with those previously obtained by the author for circular cylinders.*

The effects of applying tension to a bar by shearing stresses over its faces are considered in this connection. The correction to the readings of an extensometer (which measures the surface stretch), owing to the difference of this distribution of terminal stress from the one usually assumed, is investigated. It is found that no error will be introduced provided no measurements are taken within a distance from the grips less than one and a-half times the long diameter of the section.

Finally the possible cases of solutions in finite terms are discussed, and such a solution is obtained for a beam which carries a uniform load. It is shown that the assumptions of the usual theory of flexure

* 'Phil. Trans.,' A, vol. 198, pp. 147—233.

are in this case no longer true, but are approximately true only if the height be very small compared with the span. The correction to the curvature as calculated from the usual formula is found to be a constant.

The paper concludes with an account and a short discussion of the work of Lamé and Clapeyron, de Saint-Venant, Boussinesq, and, more recently, of M. Mathieu,* M. Ribière,† and Mr. J. H. Michell,‡ which bears upon the subject of rectangular beams. Although, in certain cases, some of the results overlap, the attempt has been made in the paper to co-ordinate them, and to present them in a more complete form, and to develop further the two-dimensional theory, so as to obtain solutions to various interesting questions relating to the effects of isolated loads.

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## "Antarctic Origin of the Tribe Schœneæ." By C. B. CLARKE, F.R.S. Received March 12,—Read April 24, 1902.

[PLATE 14.]

The map annexed to this paper is designed to illustrate the geographic distribution of all the species of the Schœneæ—a sub-ordo or tribus of the Cyperaceæ.

The result suggests a flow in geologic time of the sub-order from the South Pole up the three great southern prolongations of land, viz., Oceania, South Africa, Temperate South America; the number of species dying away rapidly as we recede from the South Pole.

I explain how the map is made. I take the outline map of the World divided into twenty-three geographic sub-areas, and my MSS. of the sub-order Schœneæ which show the distribution of every species with reference to these twenty- three sub-areas.

The first species is *Carpha alpina*, R. Br., which I see in the MS. has been collected in the sub-areas 12, 13, 14, 23. I put a spot of black paint in each of these four sub-areas, and proceed to the next species. I have treated 262 species in black dots, two in rings, two in crosses. The black dots do not signify anything as to the abundance of a species; nor in Australia and the Cape do they indicate more than that the species has been collected in that sub-area. But the outlying scattered spots in Central Africa, Japan, Jamaica, &c., are placed as accurately as the scale of the map would admit.

* 'Théorie de l'Elasticité,' Paris, 1890; also 'Comptes Rendus,' vol. 90, pp. 1272—74.

† 'Sur Divers Cas de la Flexion des Prismes Rectangles,' Bordeaux, 1889; also 'Comptes Rendus,' vol. 126, pp. 402—404 and 1190—92.

‡ 'Quart. Journ. Math.,' vol. 32.