Fig. 28.—Gold leaf on glass, exposed to mercury vapour, and then heated sufficiently to drive off the mercury.
Transmitted light, green screen.
Objective 12 mm. N.A. 0·65. Magnification × 440.
The dark patches were deep green, and the clear parts between them were colourless.

Fig. 29.—Silver film on glass, annealed by heating at 350°.
Transmitted light with green screen.
Objective 12 mm. N.A. 0·65. Magnification × 440.

Fig. 30.—The same by dark ground illumination.
(Oblique transmitted light).
Objective 16 mm. N.A. 0·3. Magnification × 440.

"The 'Hunting' of Alternating-Current Machines." By BERTRAM HOPKINSON, M.A. Communicated by Professor J. A. EWING, F.R.S. Received June 16,—Read June 18, 1903.

Many years ago the late Dr. John Hopkinson showed that if a pair of alternating-current dynamos, A and B, mechanically separate but connected electrically in parallel, be running steadily on a constant load and with a constant driving power, and if the steady motion be slightly disturbed, say by momentarily retarding A, then A will do less and B more than its share of the work, with the result that there will be a balance of force tending to accelerate A and to retard B and so to restore the state of steady motion. In other words the two machines tend to keep in step. Similar considerations apply to a synchronous alternating-current motor worked from supply mains—it tends to keep in step with the generators supplying it.

It has been found in practice that as a general rule the paralleled alternators do keep in step, but in a not inconsiderable number of cases great trouble has been caused by a tendency in the machines to develop gradually increasing oscillations about the state of steady motion in which they are in step with one another. This oscillation or "hunting" leads to violent cross magnetising currents, and sometimes the machines drop out of step altogether. This phenomenon has received a great deal of attention from the practical side, the object being of course to put an end to it. This experimental study has resulted in empirical rules as to fly-wheel effect, and in the various damping devices or "Amortisseurs" which are now largely used on alternating-current machinery and generally give satisfactory parallel running.

Theoretical treatment of hunting has been confined (so far as I am

aware) to the work of Kapp, and of others on similar lines,* who have ascribed it to resonance. Take the case of a synchronous alternating current motor driven from a source of alternating E.M.F. of constant amplitude and periodicity. In all essentials this is the same problem as that of one alternating machine running in parallel with a number of others, but the results are simpler to express. The motor is supposed to work against a constant resistance, and corresponding to that resistance there will be a certain state of steady motion in which the motor runs with a constant speed equal to that of the generators supplying it and with a constant lag e behind them, and develops a torque T corresponding to the external resistance. If the angle of lag be increased to $e+\xi$, the torque will be increased to $\mathrm{T}+\frac{d\mathrm{T}}{de}\xi$, and if the external work done by the machine remains the same there will be a force $\xi\frac{d\mathrm{T}}{de}$ or $c\xi$ tending to accelerate the motor. Kapp's argument then is that the equation of motion of the motor is $\mathrm{M}\ddot{\xi}+c\xi=0$, where M represents on a suitable scale the moment of inertia of the motor, from which it follows that it executes simple harmonic oscillations of constant amplitude and period $2\pi\sqrt{(\mathrm{M}/c)}$ about the state of steady motion. It is easy to calculate c approximately from a knowledge of the magnetic properties of the machine. Kapp worked out the period and found it to agree fairly well with observation in certain cases. He ascribed hunting to approximate equality between the period of free oscillation and that of some variation in the turning moment of the engine. Such equality would of course give rise to forced oscillations quite out of proportion to the cause. It was stated in support of this explanation that in certain cases an increase in the fly-wheel effect of the machine—viz., an increase in the period of oscillation—was found to aggravate the evil, contrary to what would at first sight be expected. And indeed it is probable that some cases of hunting are due to resonance.

I believe, however, that there have been cases in which it has been difficult or impossible to discover any external disturbing cause of approximately the same period as that of the oscillation. One case of the kind has come under my notice. A small single-phase synchronous motor, to be presently described, hunted violently under certain conditions when worked off the Wimbledon supply mains. The period of the oscillations could be varied continuously from about 0·35 to 0·45 second by appropriate variation of the self-induction in series with the motor. Furthermore, the hunting occurred equally, and with the same period, with either of two different generators working in the Power Station, one being three times the size of the other. It was

* See 'Dynamomaschinen für Gleich- und Wechselstrom,' von Gisbert Kapp, p. 401; also a paper by Hans Görges, 'Elektrotechnische Zeitschrift,' vol. 8, 1900.

clear, therefore, that in this case at any rate the hunting was not due to resonance but to some essential instability in the motion of the motor itself.

It is easy to see how such instability could arise. In the argument given above it has been assumed that the torque is dependent only on the relative position of motor and generator and not on their relative velocity. As a matter of fact there is a term in the torque dependent on the velocity, and the equation of motion is $M\ddot{\xi}+b\dot{\xi}+c\xi = 0$. Higher differential coefficients than the second may and in fact do come in, but these are the most important terms as a rule. The solution of this equation is, if b is small,

$$\xi = \xi_0 e^{-b/2M} \sin\left[\sqrt{\left(\frac{c}{M}\right)}\,t+\eta\right].$$

If b be positive, the amplitude of the oscillations continually decreases. If, however, b were negative, even though very small, the oscillations would continually increase and the motion be essentially unstable. Most dynamical systems are affected with viscosity, in which case b is positive, but systems are not unknown in which the contrary is the case. Watt's Governor is such a system.* Its oscillations about steady motion are given by a cubic equation, the two complex roots of which have positive real parts and correspond to constantly increasing oscillations. There is no doubt that the motion of a synchronous motor is under certain conditions another instance of the same thing.

Suppose for the present that the motor has a permanent magnet or saturated field, and that it is working against a constant load.

Let θ be an angle defining the position of the armature in space, $I = A \sin\theta$ the induction linked with the field coils and with the armature when in position θ. In virtue of the above assumption, A is constant. Let t be the time, and $E\cos pt+F\sin pt$, the E.M.F. of the source of supply; L the self-induction and ρ the resistance of the armature and any conductors in series with it.

Assuming for the moment that the motor is moving with uniform angular velocity, let $u = \alpha_0 \sin pt+\beta_0 \cos pt$ be the current in the armature. The epoch of t is as yet unchosen; choose it so that in the steady motion $\theta = pt$. Now suppose that the state of steady motion is slightly disturbed so that the motor oscillates about it. Then we have in the disturbed motion:

$$\theta = pt+\xi \quad \text{and} \quad u = (\alpha_0+\alpha)\sin pt+(\beta_0+\beta)\cos pt,$$

where ξ, α, β are small quantities varying periodically with the time. Experience shows that in all cases the period of the variation is long

* See Routh's 'Rigid Dynamics,' vol. 2 (1892), p. 74; also Maxwell's Collected Papers, vol. 2, p. 105.

compared with $2\pi/p$ the period of the alternating current. The external E.M.F. remains the same—$E\cos pt + F\sin pt$—the induction $A\sin\theta$ is also undisturbed by the oscillation.

Forming the equation for the E.M.F. at the terminals of the motor in the usual way and equating it to the impressed E.M.F. we find:

$$
\begin{aligned}
E\cos pt + F\sin pt &= \rho u + L\dot{u} + \frac{dI}{dt} \\
&= \rho\{(\alpha_0+\alpha)\sin pt + (\beta_0+\beta)\cos pt\} \\
&\quad + Lp\{(\alpha_0+\alpha)\cos pt - (\beta_0+\beta)\sin pt)\} \\
&\quad + L\dot{\alpha}\sin pt + L\dot{\beta}\cos pt \\
&\quad + A(p+\dot{\xi})\cos(pt+\xi).
\end{aligned}
$$

This equation is rigorously accurate under the assumptions proposed. Now equate the coefficients of $\cos pt$ and $\sin pt$ on the two sides, neglect products of the small quantities $\alpha, \beta, \xi, \dot{\alpha}$, etc., and separate the large terms corresponding to steady motion and the small terms corresponding to disturbed motion in the usual way.

Thus for steady motion

$$E = \rho\beta_0 + Lp\alpha_0 + Ap, \qquad F = \rho\alpha_0 - Lp\beta_0,$$

and for the disturbed motion

$$L\dot{\alpha} + \rho\alpha - Lp\beta - Ap\xi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$Lp\alpha + L\dot{\beta} + \rho\beta + A\dot{\xi} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The torque developed by the motor is

$$
\begin{aligned}
u\frac{dI}{d\theta} &= A\cos\theta\{(\alpha_0+\alpha)\sin pt + (\beta_0+\beta)\cos pt\} \\
&= A\cos(pt+\xi)\{(\alpha_0+\alpha)\sin pt + (\beta_0+\beta)\cos pt\} \\
&= \tfrac{1}{2}(A\beta_0 + A\beta - A\alpha_0\xi) + \text{terms of period } \pi/p \text{ and quicker period},
\end{aligned}
$$

again neglecting products of the small quantities. The large part of this, $\frac{1}{2}A\beta_0$, is equal to the constant resistance, the small balance is available for accelerating the motor. Thus the third equation is obtained:

$$2M\ddot{\xi} + A\alpha_0\xi - A\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

To solve equations (1), (2), and (3), we write as usual $\alpha = Pe^{xt}$, $\beta = Qe^{xt}$, and $\xi = Re^{xt}$, and so get, after substitution and elimination of P, Q, and R,

$$\begin{vmatrix} Lx+\rho & -Lp & -Ap \\ Lp & Lx+\rho & Ax \\ 0 & A & -(A\alpha_0+2Mx^2) \end{vmatrix} = 0,$$

which reduces to

$$(A\alpha_0+2Mx^2)\{(Lx+\rho)^2+L^2p^2\}+A^2L(x^2+p^2)+A^2\rho x = 0 \ldots (4)$$

The condition of stability of motion is, as explained in works on dynamics, that the real roots of this equation shall be negative, and that the real parts of the complex roots shall be negative. The criterion for this can be written down,* but in this case it is simpler to proceed by approximation. It is known that x is a small quantity, the period of the oscillation we are investigating being in any practical case at least fifteen times that of the alternating current. Neglect x/p and $x\rho/Lp^2$ therefore altogether in the first instance. Thus we obtain

$$(A\alpha_0+2Mx^2)(L^2p^2+\rho^2)+A^2Lp^2 = 0,$$

whence

$$x = i\sqrt{\left\{\frac{1}{2M}\left(A\alpha_0+\frac{A^2Lp^2}{L^2p^2+\rho^2}\right)\right\}} = i\delta, \text{ suppose.}$$

so that to this order of approximation the motor executes simple harmonic oscillations of constant amplitude and period $2\pi/\delta$. This, subject to an approximation which holds good in practical cases, is the result obtained by Kapp. Now, suppose that $x = \gamma+i\delta$, and substitute in the original equation (4). Neglect γ^2, x^2/p^2, and γ/p, but keep x/p, $x\rho/Lp^2$, etc.

The result is

$$4M\gamma i\delta(L^2p^2+\rho^2)+(A\alpha_0-2M\delta^2)\,2Li\delta\rho+A^2\rho i\delta = 0,$$

* The condition that the real roots and the real parts of the complex roots of the biquadratic

$$ax^4+bx^3+cx^2+dx+e = 0$$

shall all be negative is that the quantity

$$bcd-ad^2-eb^2,$$

and the coefficients a, b, c, d, e shall all have the same sign (see Routh's 'Rigid Dynamics' (1892), vol. 2, p. 192). In the case of the biquadratic here treated, this condition reduces to

$$2A(2L\alpha_0+A)-\frac{8M}{L}(L^2p^2-\rho^2) > 0,$$

which is equivalent to the condition $Lp < \rho$ found later on, if, as is the fact, the first term can be neglected in comparison with the second.

whence
$$4M\gamma = -\frac{A^2\rho + (A\alpha_0 - 2M\delta^2)\,2L\rho}{L^2p^2+\rho^2}$$
$$= -\frac{\rho}{L^2p^2+\rho^2}\left(A^2 - \frac{A^2Lp^2 2L}{L^2p^2+\rho^2}\right)$$
$$= \rho A^2\,\frac{L^2p^2-\rho^2}{(L^2p^2+\rho^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

If, therefore, $Lp > \rho$, as is nearly always the case, γ is positive, and the motion is unstable, the oscillations if once started continually increasing in amplitude according to the law $e^{\gamma t}$.

In the above it has been assumed that the field is unaffected by the oscillations, remaining the same in the disturbed as in the steady motion. As a matter of fact this is not usually the case, the field is disturbed by the oscillation owing to the varying armature reaction. The general effect is easily expressed thus. The induction linked with armature and field coils in the steady motion is supposed, as before, to be A sin θ. In the disturbed motion the field is slightly altered in amount, and slightly distorted, in a periodic way, and becomes $I = (A+a)\sin\theta + b\cos\theta$, where a and b are small quantities dependent on α, β, and ξ. Terms involving $\sin 2\theta$ and $\cos 2\theta$ will also appear to some extent, but may for practical purposes be neglected. The current in the armature is

$$u = (\alpha_0+\alpha)\sin pt + (\beta_0+\beta)\cos pt$$
$$= (\alpha_0+\alpha)\sin(\theta-\xi) + (\beta_0+\beta)\cos(\theta-\xi)$$
$$= \alpha_0\sin\theta + \beta_0\cos\theta$$
$$+ (\alpha+\beta_0\xi)\sin\theta + (\beta-\alpha_0\xi)\cos\theta,$$

and the general effect of the varying current on the field is that

$$a = \kappa(\alpha+\beta_0\xi), \qquad b = \lambda(\beta-\alpha_0\xi),$$

where κ and λ are constant for any given state of steady motion. κ represents the change in total induction produced by varying the "wattless" component of the current, and λ the change in distortion produced by varying the other component. Taking these expressions for the induction, it is easy to find the time of oscillation and rate of increase as in the simpler case investigated above. It will suffice here to give the results. The period of the oscillation is $2\pi/\delta$, where

$$2M\delta^2 = +A\alpha_0 - \lambda\alpha_0^2 - \kappa\beta_0^2 + \frac{\{(A-\lambda\alpha_0)^2(L+\kappa) + \kappa^2\beta_0^2(L+\lambda)\}\,p^2}{(L+\kappa)(L+\lambda)\,p^2+\rho^2},$$

and the amplitude increases at the rate $e^{\gamma t}$, where

$$4M\gamma = \frac{\rho}{\{(L+\kappa)(L+\lambda)\,p^2+\rho^2\}^2}\left[p^2\{(A-\lambda\alpha_0)^2(L+\kappa)^2 + \kappa^2\beta_0^2(L+\lambda)^2\}\right.$$
$$\left. - \rho^2\{(A-\lambda\alpha_0)^2+\kappa^2\beta_0^2\}\right] \ldots\ldots\ldots \quad (6).$$

As a rule $\kappa\beta_0$ is small compared to $(A - \lambda\alpha^0)$, so that the criterion of instability in this case is approximately $(L+\kappa)\,p>\rho$, and i generally fulfilled. The value of $(L+\kappa)$, which will not vary greatly with load, may be determined by observing the current given by the machine when short-circuited, including, of course, in its circuit all resistances and self-induction up to the constant source of E.M.F., which has been postulated. The apparent resistance under these circumstances is approximately $\sqrt{([L+\kappa]^2p^2+\rho^2)}$. On the other hand, the value of L may be approximately obtained by observing the current when the machine is at rest, and connected to the mains, the field coils being short-circuited. The apparent resistance of the whole circuit under these circumstances is $\sqrt{([L^2p^2+\rho^2])}$. If the motor be under-excited α_0 is positive. β_0 is positive for a motor and negative for a generator. The E.M.F. $Ap \cos pt$ is equal to the impressed E.M.F. less that required to drive the current through the self-induction and resistance of the circuit; hence, unless Lp, or ρ, or the current, be large, $Ap/\sqrt{(2)}$ is not much different from the applied E.M.F. [$\sqrt{(\text{mean}^2)}$]. By reducing the exciting current Ap is somewhat reduced, owing to the increased armature current; but the reduction is far from being in proportion to the decrease in exciting current. On machines with large armature reaction and small self-induction, Ap is practically constant, and equal to $\sqrt{(2)}$ times the impressed E.M.F. [$\sqrt{(\text{mean}^2)}$].

Damping Coils.

In practice there are, of course, many causes unconsidered in the above investigation which tend to damp out the oscillations, and which in all but exceptional cases overpower the forces making for instability, and render the motion stable. Air resistance and local currents in the armature give rise to forces of this nature, which increase roughly in proportion to the velocity and so appear in the equations of motion as true viscous terms. The most important damping effect, however, is that due to the variations of the field of the motor, which, as stated in the last paragraph, are generally produced by the oscillation. These variations give rise to periodically varying electric forces in the substance of the magnets and in the field coils, which cause currents in the former and variations of the current in the latter. These induced currents re-act on the armature, producing changes in the torque which tend to damp out the oscillations. In many machines the effect is intensified by putting additional damping coils of copper or "Amortisseurs," as they are called, round the field magnets. I propose shortly to investigate this damping effect.

The general effect of the induced currents in the magnets and in

their surrounding coils (whether damping coils or field coils)* is equivalent to that of a circuit of a certain resistance R supposed to surround completely laminated magnets of the same size and shape magnetised by a constant field current. The induction linked with this circuit will be, taking the notation already used, ν I or ν $(A+a)$, where ν is a constant. The current round the circuit, produced by the oscillation, is, therefore, $-\frac{\nu}{R}\frac{dI}{dt}$, or $-\frac{\nu}{R}\frac{da}{dt}$. This current tends to slightly demagnetise the magnets if a be increasing, and its effect on the induction linked with the field coils and armature may be represented by a term $-\mu\frac{da}{dt}$, where μ is a positive constant. The quantity μ is a time, it is the time in which the induction in the magnets falls to $1/e$ of its initial value if the field coils be suddenly short-circuited. If the armature be forced to make small oscillations, given by $\xi=\xi_0 \sin \delta t$, about the state of steady motion, the induction will be as before (p. 240).

$$I = (A+a)\sin\theta + b\cos\theta,$$

where now, however,

$$a = -\mu\frac{da}{dt} + \kappa(\alpha+\beta_0\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

while, as before

$$b = \lambda(\beta-\alpha_0\xi).$$

The E.M.F. at the terminals of the motor is

$$\begin{aligned}
E\cos pt + F\sin pt &= \rho u + L\dot{u} + \frac{dI}{dt} \\
&= \rho\{(\alpha_0+\alpha)\sin pt + (\beta_0+\beta)\cos pt\} \\
&\quad + Lp\{(\alpha_0+\alpha)\cos pt - (\beta_0+\beta)\sin pt\} \\
&\quad + (A+a)p\cos(pt+\xi) \\
&\quad - \lambda(\beta-\alpha_0\xi)p\sin pt \\
&\quad + \Big[L\dot{\alpha}\sin pt + L\dot{\beta}\cos pt + A\dot{\xi}\cos pt \\
&\quad + \frac{da}{dt}\sin pt + \lambda(\dot{\beta}-\alpha_0\dot{\xi})\cos pt\Big].
\end{aligned}$$

We drop the terms in square brackets, and investigate the damping effect of the "Amortisseur" coil separately as a small effect of the

* It is worth noting here that the result, so far as damping is concerned, is the same whether the additional copper is put into a separate short-circuited winding or into the field coils. In the latter form it assists in reducing the ordinary losses in those coils.

same order of magnitude as the opposite effect which is dependent on those terms. The result is:

$$\rho\alpha - (\mathrm{L}+\lambda)\, p\beta - \mathrm{A}p\xi + \lambda\alpha_0 p\xi = 0,$$

and

$$ap + \mathrm{L}p\alpha + \rho\beta = 0,$$

whence

$$\{\mathrm{L}(\mathrm{L}+\lambda)\, p^2 + \rho^2\}\, \beta + \mathrm{L}p^2 (\mathrm{A} - \lambda\alpha_0)\, \xi + ap\rho = 0,$$

and

$$\{\mathrm{L}(\mathrm{L}+\lambda)\, p^2 + \rho^2\}\, \alpha + a\, (\mathrm{L}+\lambda)\, p^2 - p\rho\, (\mathrm{A} - \lambda\alpha_0)\, \xi = 0.$$

These give α and β in terms of ξ and a, and substituting in (7), we find

$$a\left\{1 + \frac{\kappa\,(\mathrm{L}+\lambda)\, p^2}{\mathrm{L}\,(\mathrm{L}+\lambda)p^2 + \rho^2}\right\} + \mu\frac{da}{dt} = \xi_0 \sin\delta t \left\{\kappa\beta_0 + \frac{\kappa p\rho\,(\mathrm{A} - \lambda\alpha_0)}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2}\right\},$$

whence it follows that

$$a = a_0 \sin(\delta t + \epsilon),$$

where

$$\left.\begin{aligned} \tan\epsilon &= -\frac{\mu\delta}{1 + \dfrac{\kappa\,(\mathrm{L}+\lambda)\, p^2}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2\rho^2}} \\ \text{and}\quad a_0\left\{1 + \frac{\kappa\,(\mathrm{L}+\lambda)\, p^2}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2}\right\} &= \kappa\left\{\beta_0 + \frac{p\rho\,(\mathrm{A} - \lambda\alpha_0)}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2}\right\} \xi_0 \cos\epsilon \end{aligned}\right\} \quad (8).$$

The small periodic term in the torque, produced by the oscillation, is found, as before, to be

$$\tfrac{1}{2}\,\{a\beta_0 + (\mathrm{A} - \lambda\alpha_0)\,\beta - (\mathrm{A} - \lambda\alpha_0)\,\alpha_0\xi\}.$$

The value of this when $t = 0$ or $\xi = 0$, that is when the motor is in the position of steady motion, though not moving with the steady velocity, is

$$\tfrac{1}{2}\left\{a_0\beta_0 \sin\epsilon - (\mathrm{A} - \lambda\alpha_0)\, \frac{p\rho a_0 \sin\epsilon}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2}\right\},$$

which from (8) is equal to

$$-\frac{1}{2}\, \frac{\kappa\xi_0\mu\delta\left[\beta_0{}^2 - \dfrac{p^2\rho^2\,(\mathrm{A} - \lambda\alpha_0)^2}{\{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2\}^2}\right]}{\mu^2\delta^2 + \left\{1 + \dfrac{\kappa\,(\mathrm{L}+\lambda)\, p^2}{\mathrm{L}\,(\mathrm{L}+\lambda)\, p^2 + \rho^2}\right\}^2} \quad \ldots\ldots\ldots\ldots \quad (9).$$

The velocity at this moment is $\left[\dfrac{d}{dt}\ (\xi_0 \sin\delta t)\right]_{t=0}$ or $\xi_0\,\delta$.

The oscillation about the state of steady motion is, therefore,

resisted by a force proportional to the excess of the actual velocity over that of the steady motion, or by a true viscous force, and the magnitude of the force is, in the simple case when $\rho = 0$,

$$-\frac{1}{2}\frac{\kappa\mu\beta_0^2}{\mu^2\delta^2 + (1+\kappa/\mathrm{L})^2}\dot{\xi} = -2\mathrm{M}\gamma'\dot{\xi} \quad \ldots\ldots\ldots\ldots \quad (10).$$

Taking this force into consideration, as well as the similar force of opposite sign whose existence was proved in the first paragraph, we find finally that the free oscillations of the system are given by the equation:

$$\xi = \xi_0 e^{(\gamma-\gamma')t} \sin \delta t,$$

where γ and γ' have the following values nearly, if ρ is small,

$$\left.\begin{aligned} \gamma &= \frac{\rho \mathrm{A}^2}{4\mathrm{M}\,(\mathrm{L}+\kappa)^2\,p^2} \\ \gamma' &= \frac{\kappa\mu\beta_0^2}{4\mathrm{M}\left\{\mu^2\delta^2 + \left(1+\frac{\kappa}{\mathrm{L}}\right)^2\right\}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11).$$

These results have been obtained for a single-phase motor, with a constant self-induction and a sine-wave E.M.F. A generator running in parallel with a number of others is, of course, covered by the same equations. The results are applicable, with slight modification, to a two-phase or three-phase machine, and it would be possible, if worth while, to find the alterations introduced by the varying self-induction and distorted wave-forms which exist more or less in all actual dynamos. In no case, however, could accurate quantitative results be arrived at without great labour, for the forces here investigated are small, and in actual work a great many small disturbing causes would have to be taken into account (such, for example, as small changes in the resistance overcome by the motor)* before an accurate quantitative criterion of stability could be arrived at. A good deal of valuable information can, however, be got out of the equations as to the general effect upon stability of running of varying the constants of the machine.

(1) γ' diminishes as δ increases or the damping increases with the period of the oscillation. Hence increased fly-wheel effect always results in better damping owing to the increase in the period. Increasing the self-induction has the same effect; and it also works in favour of stability by diminishing γ. A mere alteration of M without altering δ has no effect because it alters γ and γ' in the same ratio.

(2) The damping is proportional to $\kappa\,\beta_0^2$. Now the steady torque is

* A case of great practical importance, in which the changes of load can be calculated easily, is that of the rotary converter.

$\frac{1}{2}$ A β_0. Hence for a fixed field excitation the damping term is about proportional to the square of the load; the more the machines are loaded (within limits) the better they run in parallel. This is in accord with experience. Furthermore, with constant load, β_0 is inversely proportional to A, and a reduction of A results in better damping. This I have also found to be true on the machine with which I experimented, viz.: that if with a constant load you diminish the field current you get more stable running.

(3) The co-efficient of instability, or γ, will in most cases (for which ρ is small compared with Lp) be proportional to ρA^2 and inversely proportional to $L^2 p^2$.

(4) If μ be increased from zero, in other words if the resistance to the induced currents in and about the pole pieces be diminished, the damping effect corresponding to γ' first increases to a maximum and then diminishes. It is possible therefore to carry the application of "Amortisseurs" too far. That this is so is obvious when one considers that a coil of no resistance round the pole piece would completely destroy all variation of induction and all the damping effect which depends on such variations. It would in fact correspond to the case first investigated of a motor with a fixed field, or without armature reaction, which as shown is usually unstable.

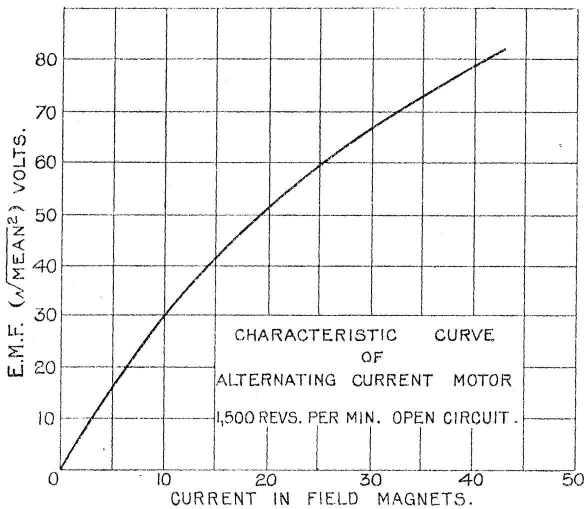
(5) Referring to the general expression for γ' (equation 9) in which the resistance is taken into account, it appears that if $\beta_o{}^2 < \frac{p^2\,\rho^2\,(A - \lambda\alpha_o)}{\{L\,(L + \lambda)\,p^2 + \rho^2\}^2}$, γ' becomes negative, and the damping coils, instead of reducing the oscillations, actually tend to increase them. $\frac{(A - \lambda\alpha_0)\,p}{\sqrt{L\,(L + \lambda)\,p^2 + \rho^2}}$ is about equal to C($\sqrt{2}$), where C is the current in the machine when standing and connected to the mains. The limits outside which β_0 must lie in order that the damping coils may operate as such, and not as additional causes of instability, are therefore about $\pm \frac{\rho}{Lp}$ C($\sqrt{2}$), and are very narrow in machines of any size. The matter only becomes practically important in the case of small machines, and of motors on long transmission lines. The behaviour of the small motor with which I experimented is considerably influenced in this way.

Experimental Confirmation.

I have in a general way confirmed the results here obtained by experiment on a small alternating current motor. This machine is a 4-pole generator, made by the Westinghouse Company, and intended to give an output of about 10 ampères continuous current at 110 volts. It was converted into a synchronous alternating current machine by fitting slip rings on to it.

The machine was separately excited, and the no-load characteristic curve for a speed of 1500 revolutions per minute is shown on fig. 1.

Fig. 1.

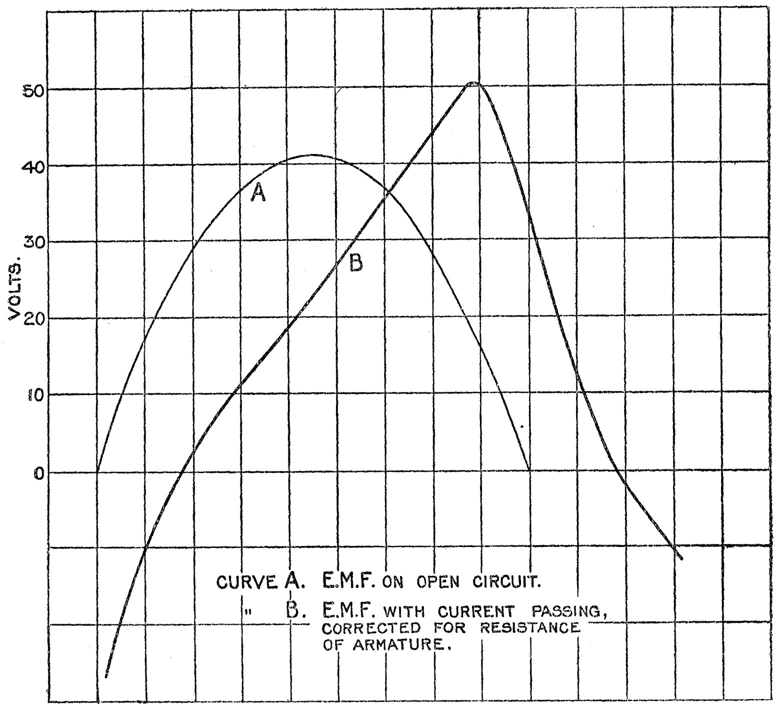


The currents are in arbitrary units, the potential is the square root of mean square of the alternating potential. The lower parts of the characteristic are a good deal affected by the previous treatment of the magnets; the curve gives average values. The machine gives, at no-load and at all excitations, very approximately a sine-wave of E.M.F., the actual value being E ($\sin pt + \frac{1}{30}\sin 3\,pt +$ terms of higher order). With a heavy non-inductive load and low field, however, the field is a good deal distorted. In fig. 2 is shown the curve of E.M.F. of the machine when excited with a current of 10·5 and working as a generator on a non-inductive resistance amounting together with the armature resistance to 1·5 ohms.

The machine was worked off the local supply mains, which gave a potential of from 94—98 volts $\sqrt{(\text{mean}^2)}$ with a frequency of 50 per second. The impressed E.M.F. is also approximately a sine-curve. The resistance and reactance (Lp) of the transformer, mains to the power house, &c., were determined from the drop in pressure at the terminals when giving a heavy non-inductive and inductive current respectively. The resistance was equivalent to about 0·1 ohm in the motor circuit, the reactance to about 0·12 ohm. The pole-pieces of the machine are laminated and cast into a solid yoke.

The pole pieces are of $2\frac{1}{2}$-inch square section, and $3\frac{1}{2}$ inches deep from the pole face to where they are set into the yoke. The diameter of the armature is $5\frac{1}{2}$ inches.

Fig. 2.



Rough determinations were made of the various constants of the machine; the resistance measured in the ordinary way was found to be 0·55 ohm cold. In normal working the resistance averaged about 0·7 ohm, making a total of 0·8 ohm with the resistance in the mains. The self-induction L, or that part of the induction which is linked with the armature but not with the field coils, was measured by observing the drop of potential across the armature when passing an alternating current, the field coils being short-circuited so as to eliminate the induction linked with them. The apparent resistance was assumed to be $\sqrt{(L^2p^2+\rho^2)}$. The result was that Lp varied from 0·65—0·93 ohm, according to the position of the armature. We take 0·8 ohm as the mean value of Lp, making 0·92 ohm total reactance with the reactance in the mains. For the determination of κ the machine was run as a generator on various non-inductive resistances varying from nothing up to 3 ohms. It was assumed that if V be the open circuit potential, C the current [$\sqrt{(\text{mean}^2)}$ in each case] and R the total resistance including armature, then $V=C\sqrt{[(L+\kappa)^2p^2+R^2]}$. So determined the value of $(L+\kappa)p$ was found to range from 1·3 ohms in the first case (short circuit) to 1·1 ohms in the second.

This method of measurement is only strictly accurate if $\kappa = \lambda$, in which case the value of κ should appear constant for all resistances. For present purposes we may take the higher value 1·3 ohms (which is the more nearly accurate) as a good enough approximation for $(L + \kappa)p$, or 1·42 ohms if the additional reactance of the mains is included, and 0·5 ohm for κp.

The power required to drive the motor unloaded against friction, hysteresis, etc. (taken as the applied watts less the C^2R losses in the armature), was found to average about 250 watts. This is the value of $\frac{1}{2}Ap\beta_0$ in all cases when the motor is described as unloaded.

Finally the value of μ was roughly determined in the following way:—The current was passed through the field coils, the machine being at rest and the circuit of the armature open. The field coils were short-circuited at a definite instant of time, and the current in them was measured at various times after the short-circuit took place. The short-circuiting was effected by means of a long heavy pendulum which closed a switch at the lowest point of its swing. The pendulum carried a contact maker which at a determinate later point in the swing connected a resistance included in the field circuit to an electrometer. The fall of potential in the resistance gave the current at the moment of making contact, and the time could be calculated from the length of the pendulum swing between closing the switch and making contact. It was found that with sufficient accuracy for present purposes the value of the current in the field magnets was halved in 0·07 second. Since in this case the current diminishes according to the law $e^{-t/\mu}$, it follows that μ is about one-tenth of a second; μ varies somewhat with the temperature of the field coils, etc., one-tenth can be taken as its order of magnitude.

Taking the above values of the constants, it appears that with the motor connected direct to the mains γ is negative. Furthermore, unless the machine is very much under-excited, we have

$$\frac{p\rho\,(A - \lambda\alpha_0)}{L\,(L + \lambda)\,p^2 + \rho^2} = 30\sqrt{(2)}\text{ ampères, roughly,}$$

taking 60 $\sqrt{(2)}$ volts as a rough value for $(A - \lambda\alpha_0)\,p$, ρ as 0·8 ohm, and $\sqrt{[L\,(L + \lambda)]}\,p$ as 1 ohm, so that γ' is also negative unless β_0 has a value comparable with 30 $\sqrt{(2)}$ ampères. We should, therefore, expect this machine to be violently unstable. This was in fact, found to be the case; when connected direct to the mains as a motor with an exciting current of thirty or more the machine very rapidly developed oscillations, and finally dropped out of step altogether. The oscillations occurred just the same, whether a large or small generator was working at the Power Station. They were not affected by putting heavy rings of copper round the pole-pieces in addition to the field winding. It was only by reducing the field current to 13·5 and loading the machine

with a brake until the applied power was about 1800 watts that the "hunting" could be stopped.

[That the hunting in this case was principally due to the negative value of γ' was shown by the fact that it was not possible to stop the hunting by the introduction of external resistance. Even with a resistance amounting to 2·5 ohms, with which ρ would certainly be greater than $(L+\kappa)\,p$, the motion was still unstable. The impossibility of making the motion stable by adding resistance puzzled me considerably at first, as I had not noticed the importance of the negative term in γ', but had treated it as a negligible quantity—*June* 28th.]

By inserting self-induction in series with the motor the violence of the "hunting" could be gradually diminished, and at the same time its period could be continuously increased from about 0·35 second to 0·45 second. When the self-induction was such that Lp amounted to about 6 ohms the motion became stable, the oscillations when once started rapidly dying away. This is in accordance with conclusion (1) above, and the continuous variation of the period though the hunting continues is proof of the essential instability of the motion.

With external self-induction giving a total reactance of about 3 ohms and resistance of 1·2 ohms the stability depended on the load and on the exciting current. The following observations were made:—

(*a*) Exciting current 11·0, motor unloaded. The motion was stable, oscillations if started dying slowly away. Period of oscillation 0·42 second.

(*b*) Exciting current was increased to 18·5 when the motion became unstable. By loading the motor slightly with a brake the "hunting" could be stopped. Period 0·38 second.

(*c*) Exciting current increased to 30·5. The motion with the motor unloaded became violently unstable. By loading the motor with a brake until the power delivered to motor (excluding external self-induction) amounted to 1000 watts, the "hunting" could be stopped. It was curious to watch the effect of the load on the current taken by the machine. The ammeter had such a slow period that it could not follow the variations in current due to the oscillations, but remained steady, giving the average of the square of the current. When the machine was "hunting" the ammeter showed a current of about 27 ampères, and the wattmeter an applied power of about 710 watts, two-thirds of which were, of course, accounted for by the C^2R losses. On putting the load on, the current as shown by the ammeter gradually diminished until when the motion was steady it was 13·5 ampères. The power indicated by the wattmeter was now again about 1000 watts, but this time only about one-eighth was going in C^2R losses.

The last series of experiments is a good confirmation of conclusions

(2) and (5) above. It is worth while in this case, the conditions of which approximate to those assumed in the theory, to determine the order of magnitude of γ and γ'.

The following measurements were made :—

Total reactance (Lp) varied between 2·81 and 3·18 ohms, according to position of armature. Its value may be taken as constant, and equal to 3 ohms without serious error.

Total resistance (ρ) = 1·2 ohms.

In case (c) the potential across the motor terminals (that is excluding the external self-induction) was 75 volts. Hence, Ap is less than $75\sqrt{(2)}$ volts. On the other hand, Ap is greater than $\sqrt{(2)}$ times the open-circuit potential of the machine with exciting current 30·5, which was measured and found to be 65 volts. Take, therefore Ap as equal to $70\sqrt{(2)}$. The power supplied to motor terminals (again excluding the external self-induction) was 1000 watts. The current was 13·5 ampères, hence the C^2R loss in armature is 110 watts (armature resistance taken as 0·6 ohm), and we have

$$\tfrac{1}{2}Ap\beta_0 = 890 \text{ watts},$$

and

$$\beta_0 = 12{\cdot}7\sqrt{(2)} \text{ ampères}.$$

Thus we obtain

$$4M\gamma p^2 = \frac{\rho A^2 p^2}{(L+\kappa)^2 p^2} = \frac{1{\cdot}2 \times (70\sqrt{2})^2}{(3{\cdot}5)^2} = 960.$$

(See formula 11 above.)

For γ' the more accurate formula (9) must be taken, since, as in all experiments with this motor, the resistance term is important. The formula may be written with sufficient accuracy

$$4M\gamma' p^2 = \frac{\kappa\mu p^2\left[\beta_0{}^2 - \dfrac{p^2A^2\rho^2}{(L^2p^2+\rho^2)^2}\right]}{\mu^2\delta^2 + (1+\kappa/L)^2}.$$

We take $\mu = \frac{1}{10}$ sec., $2\pi/\delta$ (the period of oscillation) = 0·36 second, $\dfrac{2\pi}{p} = \frac{1}{50}$ second Hence δ is 17 and p = 314.

We have

$$\frac{Ap\rho}{L^2p^2+\rho^2} = \frac{70\sqrt{(2)} \times 1{\cdot}2}{10{\cdot}4} = 8\sqrt{(2)} \text{ ampères}.$$

Hence

$$4M\gamma' p^2 = \frac{0{\cdot}5 \times 0{\cdot}1 \times 314[(12{\cdot}7)^2 - (8)^2] \times 2}{[(0{\cdot}1)^2 \times (17)^2] + (1{\cdot}3)^2} = 680.$$

This value is, of course, very rough; the most that can be asserted is that γ' is a positive quantity of the same order of magnitude as, and

probably somewhat less than γ. The motion is only just stable with this load, however, and can be made unstable by a very small reduction of β_0, and I think the figures are a proof that there is an element of instability other than a possible negative value of γ'.

To get an approximation to the absolute value of γ, we note that

$$\delta^2 = \frac{A^2(L+\kappa)p^2}{2M\{(L+\kappa)^2p^2+\rho^2\}}, \text{ nearly}$$

Hence

$$4Mp^2 = \frac{A^2p^2(L+\kappa)p^2}{\{(L+\kappa)^2p^2+\rho^2\}\delta^2} = \frac{4\times(70)^2\times 3\cdot 5\times 314}{13\cdot 6\times(17)^2} = 5500 \text{ nearly},$$

and

$$\frac{1}{\gamma} = \frac{4Mp^2}{960} = \text{about } 5\cdot 7 \text{ secs.},$$

which is the sort of magnitude one would expect from the rate of development of the oscillations when the motor is unloaded and γ' is small or slightly negative.

Note Added June 28th, 1903.

It is useful to consider the effect of increasing the dimensions of the machine on the results here obtained. Suppose that the linear dimensions of every part except the field coils are increased n-fold, that the speed remains the same. Then:

A	becomes	n^2A,
ρ	,,	ρ/n,
L	,,	nL,
κ	,,	$n\kappa$,
M	,,	n^5M.

The rated output is multiplied by between n^3 and n^4. To get the same magnetisation we require n-times the ampère turns in the field-coils. Hence if the number of turns in the field-coils and the current density be kept the same, we require n-times the section of wire, and the wire is n-times as long. Since all the other linear dimensions of the coils and of the magnetic circuit are increased n-fold, it readily follows that μ (if the magnets are completely laminated and there are no amortisseur coils) is increased n-fold.

Taking these values, we find that

$$\delta\left[=\sqrt{\left(\frac{A^2}{2M(L+\kappa)}\right)}\right] \quad \text{becomes} \quad \frac{\delta}{n},$$

$$\gamma\left(=\frac{\rho A^2}{4M(L+\kappa)^2p^2}\right) \quad ,, \quad \frac{\gamma}{n^4}.$$

Hence $\mu\delta$ is unaltered and the damping term

$$\gamma' = \frac{\kappa\mu\beta_0^2}{4\mathrm{M}\left\{\mu^2\delta^2 + \left(1 + \frac{\kappa}{\mathrm{L}}\right)^2\right\}}$$

becomes $n\gamma'$, assuming that corresponding values of the current are as $n^2 : 1$. As a fact corresponding values of the current (that is, values which are the same fraction of the maximum rated output) are in a somewhat less ratio than $n^2 : 1$, so that γ' increases but very little with n.

It appears, therefore, that the coefficient of instability decreases very rapidly with increasing size, while the damping coefficient γ' increases somewhat. At the same time, owing to the rapid decrease of $\rho/\mathrm{L}p$, the critical value of the load at which γ' becomes negative, rapidly becomes smaller in relation to the output of the machine. It may be inferred that a machine similar to that experimented with and but very little larger, would run stably at practically all loads.

In actual practice the dimensions of a pair of adjacent poles and of a corresponding piece of armature in a section perpendicular to the axis are rarely more than two or three times those on the machine here experimented with even in big alternators; the increased output is obtained by increasing the number of poles and by increasing the length of the machine. The number of poles and the length of the machine, provided the peripheral speed remains the same, have but little effect on the quantities γ, γ' and δ. In other words the performance of a machine as regards hunting is determined almost wholly by the form and dimensions, in a section perpendicular to the axis, of a pair of poles and the corresponding bit of armature. The weight of a corresponding bit of fly-wheel (if there is one) must, of course, be added to that of the bit of armature. Thus it is quite conceivable that machines of large size, but with small armature reaction and self-induction, might be constructed in which the quantity γ would be important and the running unstable, at any rate at low loads, though if the magnetic circuit of the machine were similar to that of the one experimented on (in which the self-induction and armature reaction are pretty high), and of double its linear dimensions or more, the motion would undoubtedly be stable.