"The Advancing Front of the Train of Waves emitted by a Theoretical Hertzian Oscillator." By A. E. H. LOVE, F.R.S., Sedleian Professor of Natural Philosophy in the University of Oxford. Received May 9,—Read June 2, 1904.

[PLATES 2—6.]

The waves emitted by Hertz's oscillator have been identified with those due to a vibrating electric doublet, that is to say, to a singular point (of a certain type) of the electromagnetic equations. In air or in free æther these equations may be written in the forms

$$\left.\begin{aligned} \frac{1}{\mathrm{C}}\frac{\partial}{\partial t}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z}) &= \operatorname{curl}(\alpha,\ \beta,\ \gamma) \\ -\frac{1}{\mathrm{C}}\frac{\partial}{\partial t}(\alpha,\ \beta,\ \gamma) &= \operatorname{curl}(\mathrm{X},\ \mathrm{Y},\ \mathrm{Z}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1),$$

in which C is the velocity of radiation, (X, Y, Z) denotes the electric force measured in electrostatic units, (α, β, γ) denotes the magnetic force measured in electromagnetic units. These equations are nearly identical with those which have been used by Hertz.* They differ from the latter in that C is here written for the quantity which Hertz wrote 1/A, and they differ also in the signs of the right-hand members. The reason for the latter difference is that Hertz used a left-handed system of axes of x, y, z; but it is on many grounds more convenient to use a right-handed system, as will be done here. The field due to a variable doublet at the origin, with its axis parallel to the axis of z, is expressed by equations of the form

$$\left.\begin{aligned} (\mathrm{X},\ \mathrm{Y},\ \mathrm{Z}) &= \left(\frac{\partial^2}{\partial x\partial z},\ \frac{\partial^2}{\partial y\partial z},\ -\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{\psi(\mathrm{C}t-r)}{r} \\ (\alpha,\ \beta,\ \gamma) &= \frac{1}{\mathrm{C}}\left(\frac{\partial^2}{\partial y\partial t},\ -\frac{\partial^2}{\partial x\partial t},\ 0\right)\frac{\psi(\mathrm{C}t-r)}{r} \end{aligned}\right\} \quad \ldots \quad (2),$$

in which r denotes the distance of any point (x, y, z) from the origin, and $\psi(\mathrm{C}t)$ is the moment of the doublet at time t. In Hertz's work the function ψ is taken to be a simple harmonic function of its argument, and written in a form equivalent to $\mathrm{E}l \sin n(t - r/\mathrm{C})$. This supposition would be adequate if the vibrations were maintained,

* "Die Kräfte elektrischer Schwingungen, behandelt nach der Maxwell'schen Theorie," 'Ann. Phys. Chem.' (Wiedemann), vol. 36 (1888). Reprinted in Hertz, 'Untersuchungen ü. d. Ausbreitung d. elektrischen Kraft' (Leipzig, 1892), p. 147, and in 'Electric Waves' (English edition), p. 137. The detailed references in the text are to the pages of the English edition.

or if the damping due to radiation were very slight. The actual damping of Hertz's oscillator has been investigated experimentally by V. Bjerknes,* and shown to be very considerable. Accordingly, we ought to take for ψ a function of the form

$$\psi = Ae^{-\frac{\nu}{\lambda}(ct-r)} \sin \frac{2\pi}{\lambda}(ct - r + \epsilon) \quad\quad (3),$$

where λ is the wave-length, A a constant depending upon the amplitude of the vibrations, ϵ a constant expressing the phase, and ν a constant expressing the damping. According to the experiments of Bjerknes already cited, ν may be taken to be about 0·4 when the wave-length λ is about 10 m. The effect of the introduction of the exponential factor into the expression for ψ has been investigated in an elaborate memoir by K. Pearson and A. Lee.† In that memoir it is supposed that the fixed epoch from which time is measured is the instant at which the vibrations begin, so that, at any instant, the field expressed by (2) and (3) is confined to the region within the sphere $r = ct$. In the expression for ψ the phase-constant ϵ is omitted by these authors. They have thus tacitly assumed that ψ vanishes at the front of the advancing wave.

This front is a moving surface which is a surface of discontinuity in regard to the electric and magnetic forces. Within the surface these forces are expressed by the formulæ already written down; outside the surface they must be expressed by some other formulæ. The waves, in fact, advance either through a pre-established electrostatic or electromagnetic field of some kind, or possibly through a region of space in which there is no electric or magnetic force. Whatever view may be taken of the nature of the field outside the wave-front, definite conditions must be satisfied at this surface. These conditions are known, but they have not been applied to the problem in hand. It seems worth while to make this application, and, in particular, to ascertain the effect of these conditions in modifying the results obtained by Pearson and Lee.

Let Σ denote in general a moving surface which separates two electromagnetic fields. Then it is known that Σ moves normally to itself with the velocity c. Let (X_0, Y_0, Z_0) and $(\alpha_0, \beta_0, \gamma_0)$ denote the electric and magnetic forces on that side of Σ towards which Σ advances, let l, m, n denote the direction cosines of the normal to Σ drawn towards this side, and let (X, Y, Z) and (α, β, γ) denote the electric and magnetic forces on the other side. Then at any point of Σ it is known that the following six equations must be satisfied:—

* 'Ann. Phys. Chem.' (Wiedemann), vol. 44 (1891).

† 'Phil. Trans.,' A, vol. 193, 1900.

$$\begin{aligned} X - X_0 &= n(\beta - \beta_0) - m(\gamma - \gamma_0), & \alpha - \alpha_0 &= m(Z - Z_0) - n(Y - Y_0), \\ Y - Y_0 &= l(\gamma - \gamma_0) - n(\alpha - \alpha_0), & \beta - \beta_0 &= n(X - X_0) - l(Z - Z_0), \\ Z - Z_0 &= m(\alpha - \alpha_0) - l(\beta - \beta_0), & \gamma - \gamma_0 &= l(Y - Y_0) - m(X - X_0), \end{aligned} \qquad \ldots\ldots\ldots\ldots \quad (4).$$

These equations may be expressed in words in the statements that the components of electric and magnetic force along the normal to Σ are continuous, and that the discontinuities of the tangential components of the electric and magnetic forces are equal in magnitude and are directed along lines at right angles to each other in such a way that the discontinuity of electric force, the discontinuity of magnetic force, and the normal to the surface, in this order, are parallel to the axes of x, y, z in a right-handed system.* The case in which there is no electric or magnetic force on the side of Σ towards which it advances is included by putting (X_0, Y_0, Z_0) and $(\alpha_0, \beta_0, \gamma_0)$ equal to zero, and the case in which the field on this side of the surface is electrostatic is included by putting $(\alpha_0, \beta_0, \gamma_0)$ equal to zero.

The conditions (4) have been established by a rather troublesome process which may be replaced by the following simpler argument:— The ordinary equations (1) of the field fail at the surface of discontinuity Σ through the infinity of some of the differential coefficients $\partial x/\partial t$, ... Consider the axis of x to be parallel to the normal to Σ at a point P. Then as Σ passes over P the state of the medium at P changes from that expressed by $(X_0, \ldots \alpha_0, \ldots)$ to that expressed by $(X, \ldots, \alpha, \ldots)$. Suppose the change to take place in a very short time δt, and multiply both sides of the equations (1) by $c\delta t$. Then in the left-hand members we must write $X - X_0$ for $\dfrac{\partial X}{\partial t}\delta t$, and similarly for the other quantities of the same kind. Again, we may put $c\delta t = \delta x$, where δx is the distance over which the small part of Σ near to P moves in the interval δt; and then the limit of $\dfrac{\partial \beta}{\partial x} c\delta t$ or $\dfrac{\partial \beta}{\partial x}\delta x$ is the difference of the values of β just before and just behind the surface Σ, or it is $\beta_0 - \beta$. The limits of such quantities as $\dfrac{\partial \alpha}{\partial y} c\delta t$, in which the differentiation is performed with respect to any co-ordinate other than x, are zero.

From the six equations (1) we deduce in this way the six equations

$$\begin{aligned} X - X_0 &= 0, & Y - Y_0 &= -(\gamma_0 - \gamma), & Z - Z_0 &= \beta_0 - \beta, \\ \alpha - \alpha_0 &= 0, & -(\beta - \beta_0) &= -(Z_0 - Z), & -(\gamma - \gamma_0) &= Y_0 - Y. \end{aligned}$$

* These results were given effectively in a paper by the author in 'Proc. London Math. Soc.' (Ser. 2), vol. 1, p. 37 (1903). Equivalent conditions appear to have been employed by O. Heaviside, 'Electrical Papers,' vol. 2, pp. 405 *et seq.*

These equations express the same relations between the forces and the direction of the normal to Σ as are expressed by (4).

In the application of conditions (4) to the problem of the Hertzian oscillator, the external field (X_0, Y_0, Z_0), $(\alpha_0, \beta_0, \gamma_0)$ is that which is established at the instant when the vibrations begin. At this instant the brass balls of the oscillator are so highly charged that the electric strength of the air between them gives way. The initial field is that due to the charges at this instant, so that it can most appropriately be represented as the electrostatic field of a fixed doublet. If B denotes the moment of this doublet, the field in question is expressed by the equations

$$\left.\begin{aligned}(X_0, Y_0, Z_0) &= \left(\frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z^2}\right)\frac{B}{r}\\ (\alpha_0, \beta_0, \gamma_0) &= 0\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots \quad (5).$$

Now write down the complete expressions for X, Y, Z, and α, β, γ, in accordance with equations (2). Denoting differential coefficients of the function ψ with respect to its argument by accents, these expressions are

$$\left.\begin{aligned}X &= \frac{xz}{r^5}(3\psi + 3r\psi' + r^2\psi''), \qquad Y = \frac{yz}{r^5}(3\psi + 3r\psi' + r^2\psi''),\\ Z &= -\frac{x^2+y^2}{r^5}(3\psi + 3r\psi' + r^2\psi'') + \frac{2}{r^3}(\psi + r\psi'),\\ \alpha &= -\frac{y}{r^3}(\psi' + r\psi''), \qquad \beta = \frac{x}{r^3}(\psi' + r\psi''), \qquad \gamma = 0,\end{aligned}\right\} \quad (6).$$

In like manner, complete expressions for X_0, Y_0, Z_0, as given by (5), are

$$X_0 = \frac{xz}{r^5}3B, \qquad Y_0 = \frac{yz}{r^5}3B, \qquad Z_0 = -\frac{x^2+y^2}{r^5}3B + \frac{2}{r^3}B, \ldots (7).$$

Let $t = 0$ be the instant when the vibrations begin. Then $r = ct$ is the equation of the surface separating at time t the field expressed by (6) from that expressed by (7), and the direction cosines l, m, n of the normal to this surface are x/r, y/r, z/r. When these values of X,... are substituted in (4) it will be found that ψ'' disappears, and that the equations (4) give

$$3(\psi - \beta) + 2r\psi' = 0, \qquad 2(\psi - B) + 2r\psi' = 0,$$

which must hold when $r = ct$, *i.e.*, when the argument of ψ is zero. Hence we must have

$$\psi(0) = B, \qquad \psi'(0) = 0.$$

Now take ψ to have the form (3). We find

$$A \sin\frac{2\pi\epsilon}{\lambda} = B, \qquad \tan\frac{2\pi\epsilon}{\lambda} = \frac{2\pi}{\nu} \ldots\ldots\ldots\ldots\ldots \quad (8).$$

The second of these equations determines ϵ and the first determines A in terms of B. It appears that B (the moment of the initial doublet) is the maximum moment of the vibrating doublet.*

It has now been shown that the waves expressed by (2) in which ψ is given by (3) *can* advance through the field expressed by (5), provided the constants A, B, ν, ϵ, are connected by the equations (8). Incidentally it has been shown that the waves expressed by (2) in which ψ is given by (3) cannot advance through a region in which there is no electric or magnetic force, and that the phase constant ϵ cannot vanish. In fact, the function ψ instead of vanishing at the front of the wave has there its numerically greatest value.

Expressions may be formed for the radial and transverse components of the electric force and for the magnetic force. The lines of electric force lie in planes through the axis of the doublet, and the lines of magnetic force are circles about that axis. The radial component R of the electric force is given by the equation

$$\mathrm{R} = \frac{2\cos\theta}{r^3}\mathrm{A}e^{-\frac{\nu}{\lambda}(\mathrm{c}t-r)}\left[\left(1-\frac{r\nu}{\lambda}\right)\sin\frac{2\pi}{\lambda}(\mathrm{c}t-r+\epsilon)\right.$$
$$\left.+\frac{2\pi r}{\lambda}\cos\frac{2\pi}{\lambda}(\mathrm{c}t-r+\epsilon)\right] \quad \ldots\ldots \quad (9),$$

when $\mathrm{c}t > r$, but when $\mathrm{c}t < r$ we have

$$\mathrm{R} = \frac{2\cos\theta}{r^3}\mathrm{A}\sin\frac{2\pi\epsilon}{\lambda} \quad \ldots\ldots \quad (10),$$

θ being the angle which a line drawn from the origin to a point at distance r makes with the axis of the doublet.

The transverse component Θ of the electric force is given by the equation

$$\Theta = \frac{\sin\theta}{r^3}\mathrm{A}e^{-\frac{\nu}{\lambda}(\mathrm{c}t-r)}\left[\left(1-\frac{r\nu}{\lambda}+\frac{r^2(\nu^2-4\pi^2)}{\lambda^2}\right)\sin\frac{2\pi}{\lambda}(\mathrm{c}t-r+\epsilon)\right.$$
$$\left.+\frac{2\pi r}{\lambda}\left(1-\frac{2r\nu}{\lambda}\right)\cos\frac{2\pi}{\lambda}(\mathrm{c}t-r+\epsilon)\right] \quad \ldots\ldots \quad (11),$$

when $\mathrm{c}t > r$, but when $\mathrm{c}t < r$ we have

$$\Theta = \frac{\sin\theta}{r^3}\mathrm{A}\sin\frac{2\pi\epsilon}{\lambda} \quad \ldots\ldots \quad (12).$$

* The result that the maximum moment of the vibrating doublet ought to be the same as the moment of the doublet existing at the instant when the vibrations begin is noted by M. Brillouin, 'Propagation de l'Électricité, Histoire et Théorie' (Paris, 1904), p. 313.

The magnetic force H is given by the equation

$$H = -\frac{\sin\theta}{r^2} Ae^{-\frac{\nu}{\lambda}(ct-r)}\left[\left(\frac{\nu}{\lambda}-\frac{r(\nu^2-4\pi^2)}{\lambda^2}\right)\sin\frac{2\pi}{\lambda}(ct-r+\epsilon)\right.$$

$$\left.-\frac{2\pi}{\lambda}\left(1-\frac{2r\nu}{\lambda}\right)\cos\frac{2\pi}{\lambda}(ct-r+\epsilon)\right] \quad\ldots\ldots\ldots \quad (13),$$

when $ct > r$, but when $ct < r$ it vanishes.

The radial electric force is continuous at the front of the wave, *i.e.*, at the surface $r = ct$. The discontinuity of the transverse component of the electric force at the front of the wave is

$$\frac{\sin\theta}{r}\frac{A}{\lambda^2}(\nu^2+4\pi^2)\sin\frac{2\pi\epsilon}{\lambda} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (14),$$

and this is equal, as it should be, to the magnetic force at the front of the wave.

The lines of electric force are the intersections of the planes through the axis of the doublet with a certain family of surfaces $Q = \text{constant}$.* If we denote by ρ the distance of a point from the axis, so that $\rho = r\sin\theta$, the quantity Q is $\rho\dfrac{\partial}{\partial\rho}\left(\dfrac{\psi}{r}\right)$, and the components of electric force parallel to the axis and at right angles to it are respectively $-\dfrac{1}{\rho}\dfrac{\partial Q}{\partial\rho}$ and $\dfrac{1}{\rho}\dfrac{\partial Q}{\partial z}$. The flux of electric force through any circle with its centre on the axis of the doublet may be expressed as $-2\pi Q$. The form of Q is given by the equations

$$Q = -\frac{\sin^2\theta}{r} Ae^{-\frac{\nu}{\lambda}(ct-r)}\left\{\left(1-\frac{r\nu}{\lambda}\right)\sin\frac{2\pi}{\lambda}(ct-r+\epsilon)\right.$$

$$\left.+\frac{2\pi r}{\lambda}\cos\frac{2\pi}{\lambda}(ct-r+\epsilon)\right\} \quad\ldots\ldots \quad (15),$$

when $ct > r$, and

$$Q = -\frac{\sin^2\theta}{r} A\sin\frac{2\pi\epsilon}{\lambda} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (16),$$

when $ct < r$. At the separating surface Q is continuous, just as the radial component R of electric force is continuous, and in fact we have

$$Q = -\tfrac{1}{2}\frac{\sin^2\theta}{\cos\theta} r^2 R \ldots\ldots\ldots\ldots\ldots\ldots \quad (17),$$

throughout the field.

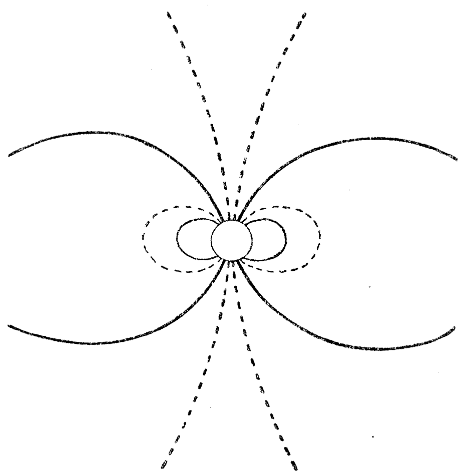
The particular case where there is no damping by radiation is included in the foregoing by putting $\nu = 0$, $2\pi\epsilon/\lambda = \frac{1}{2}\pi$; and then we have

$$\psi = A\cos\frac{2\pi}{\lambda}(ct-r), \qquad B = A.$$

* The use of the function Q in these problems was initiated by Hertz, *loc. cit.*

This is the case discussed by Hertz, but his discussion has no very definite reference to the front of the waves. The field due to the initially existing doublet is the well-known electrostatic field of a doublet, the lines of electric force being identical with the lines of magnetic force due to a magnet or to an uniformly magnetised sphere, or with the lines of flow of incompressible fluid through which a sphere is moving. These lines have been traced often,* a few of them are traced in fig. A.

FIG. A.



Hertz† has figured the lines of force within a distance of $\frac{3}{4}\lambda$ of the oscillator at a number of instants during the progress of a vibration. It would be easy to determine the modifications that ought to be made in his figures on account of the existence of a front to the advancing wave-train and of the existence outside that front of an electrostatic field. The moment of the doublet which gives rise to this field is the maximum moment of the vibrating doublet. At any instant during the vibration the electromagnetic field of the vibrator will be established within a distance from the doublet equal to the distance which light would travel in the time that has elapsed since the commencement of the vibrations. After one-eighth of a period, for instance, this field will be confined to the region within a sphere of radius $\frac{1}{8}\lambda$, and outside this sphere the field is the above-described electrostatic field. It follows that, to obtain the lines of force during the first three-quarters of a period, a circle of suitable radius should be

* See, *e.g.*, J. J. Thomson, 'Elements of Electricity and Magnetism,' p. 223, Lamb, 'Hydrodynamics,' p. 137.

† 'Electric Waves,' pp. 144, 145. The figures are reproduced by M. Brillouin, *loc. cit.*, pp. 292, 293; and by A. Gray, 'Magnetism and Electricity,' vol. 1 (London, 1898), pp. 406—408.

described round the centre of one of Hertz's figures, and the parts of his lines of force which lie outside that circle should be suppressed and their places taken by curves of the family traced in fig. A. The result that at the beginning of the vibration the moment of the doublet is a maximum would be expressed by taking his figures in the order 29, 30, 27, 28. After the electromagnetic field has become established, these figures represent the field near the vibrator at instants which are the beginning of a period, $\frac{1}{8}$ of a period later, $\frac{1}{4}$ of a period later, $\frac{3}{8}$ of a period later. If the arrow heads in his figures are reversed, they represent in the same order the field after $\frac{1}{2}$ a period, $\frac{5}{8}$ of a period, $\frac{3}{4}$ of a period, $\frac{7}{8}$ of a period from the beginning of a particular vibration, which is not the first vibration. To trace the course of the first vibration we should proceed as follows:—At the instant when the vibrations begin the field is that shown in fig. A. After $\frac{1}{8}$ of a period draw on Hertz's figure 30 a circle of radius $\frac{1}{8}\lambda$,* suppress the part of the figure outside this circle and replace it by the part of fig. A which is outside the same circle. To obtain the fields after $\frac{1}{4}$ period and $\frac{3}{8}$ period, similar work should be done upon Hertz's figures 27 and 28 with circles of radii $\frac{1}{4}\lambda$, $\frac{3}{8}\lambda$. After $\frac{1}{2}$ a period, reverse the arrow heads in Hertz's figure 29, draw on this figure a circle of radius $\frac{1}{2}\lambda$, suppress the part of the figure which is outside this circle and replace it by the part of fig. A which is outside the same circle. To obtain the fields after $\frac{5}{8}$ period, $\frac{3}{4}$ period, $\frac{7}{8}$ period, similar work should be done upon the figures 30, 27, 28 with circles of radii $\frac{5}{8}\lambda$, $\frac{3}{4}\lambda$, $\frac{7}{8}\lambda$. These modified figures have not been drawn here because a similar procedure will be adapted presently to the figures of Pearson and Lee in which account is taken of the damping by radiation.

The notation of this paper can be identified with that of Pearson and Lee by means of the equations

$$t' = t+\epsilon/\mathrm{C}, \quad \mathrm{E}l = -\mathrm{A}e^{\nu\epsilon/\lambda}, \quad 2\tau = \lambda/\mathrm{C}, \quad \chi = \frac{\pi}{2}-\frac{2\pi\epsilon}{\lambda} \quad \ldots \quad (18),$$

in which the quantities $\mathrm{E}l$, τ, χ are used by these authors,† and t' is the quantity which they denote by t. They have traced the lines of electric force, given by Q = constant, for certain chosen values of Q, for a region of space between the spheres $r = \frac{1}{10}\lambda$ and $r = \frac{5}{4}\lambda$, and for fifty-six values of t', viz.: $t' = 2\tau\,(\frac{1}{8}, \frac{1}{4}, \ldots 7)$. The chosen values of Q are such that $\mathrm{Q}\lambda/2\pi\mathrm{E}l = \pm\frac{1}{100}, \pm\frac{1}{10}, \pm\frac{3}{10}, \pm\frac{1}{2}$. The curves thus formed are shown in their Plates 1—7, each plate containing eight figures.‡ The chosen value of ν is 0·4.

* In Hertz's notation it would be $\frac{1}{4}\lambda$. Hertz has used the letter λ to denote the half wave-length.

† 2τ is the period.

‡ A number of these figures have been reproduced by M. Brillouin in plates at the end of his treatise already cited.

The lines of force drawn in some of these figures need some modification on account of the existence of a front of the wave-train. At any instant the electromagnetic field that is propagated with the waves will have reached a distance Ct from the oscillator, and therefore those parts only of the curves which lie within circles of radii $\lambda\,(t'/2\tau-\epsilon/\lambda)$ are lines of force in the actual vibrations. Outside spheres having these radii the actual field is the electrostatic field due to the fixed doublet, viz.: it is the field expressed by (5) and figured in fig. A. In the notation of (18) the lines of force in this field are given by the equation

$$\frac{Q\lambda}{2\pi El}=\frac{\sin^2\theta}{r}\frac{\lambda}{2\pi}e^{-\nu\epsilon/\lambda}\sin\frac{2\pi\epsilon}{\lambda}.$$

The continuations of the lines of force outside the wave-fronts at the various times in question are obtained by equating this expression to the values $\pm\frac{1}{100}$, $\pm\frac{1}{10}$, $\pm\frac{3}{10}$, $\pm\frac{1}{2}$. The heavy dotted, heavy continuous, fine dotted and fine continuous curves in fig. A have been drawn to correspond with these four pairs of values, λ being represented by 1 inch ($=$ 2·54 cm.).

The value of ν being 0·4, the following numerical values are found for the various quantities :—

$$\tan\frac{2\pi\epsilon}{\lambda}=5\pi,\qquad \frac{2\pi\epsilon}{\lambda}=1{\cdot}5071389,\qquad \frac{\epsilon}{\lambda}=0{\cdot}239868,$$

$$\frac{\nu\epsilon}{\lambda}=0{\cdot}0959474,\qquad e^{-\nu\epsilon/\lambda}=0{\cdot}908512,\qquad e^{-\nu\epsilon/\lambda}\sin\frac{2\pi\epsilon}{\lambda}=0{\cdot}906676,$$

$$2\pi e^{\nu\epsilon/\lambda}\operatorname{cosec}\frac{2\pi\epsilon}{\lambda}=6{\cdot}99403.$$

The circles outside which the lines of force drawn in the figures of Pearson and Lee have to be replaced by other lines are given by the equation

$$r=\lambda\,(a-0{\cdot}239868)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(19),$$

in which a has the values $\frac{1}{8}$, $\frac{1}{4}$, ... In fig. 1 of their Plate 1, r would be negative; this figure, in fact, relates to an epoch before the vibrations begin, and no part of it represents lines of force that are formed. In fig. 2 of their Plate 1, r would be about $\frac{1}{100}$ of a wave-length, so that the circle is too small to be drawn. In fig. 3 of their Plate 1, r = 0·135 of a wave-length, so that the front of the waves cannot be distinguished clearly from the inner circular boundary of the figure. Those parts only of the lines drawn in this figure which lie between the inner circular boundary and a circle of radius (0·135) λ are actual lines of force at the instant in question. Figs. 1, 2, and 3 of Plate 1 should, therefore, be omitted. In the remaining figures of Plate 1, and in figs. 9, 10 and 11 of Plate 2, parts only of the lines of force that are drawn are actual lines of force at the corresponding instants during

the vibrations. These parts lie within the circles obtained from (19) by giving to a the values $\frac{1}{2}$, $\frac{5}{8}$, ... $\frac{11}{8}$, and the corresponding instants are $t = (0{\cdot}26,\ 0{\cdot}385,\ 0{\cdot}51,\ 0{\cdot}635,\ 0{\cdot}76,\ \ldots 1{\cdot}135) \times (\text{period})$. The corresponding lines of force outside these circles are given by the equation

$$\frac{\sin^2\theta}{r} = \frac{1}{\lambda}(6{\cdot}99403)\,b \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20),$$

in which b has the values $\pm\frac{1}{100}$, $\pm\frac{1}{10}$, $\pm\frac{3}{10}$, $\pm\frac{1}{2}$. These are the curves drawn in fig. A above. The remaining figures (12—56) of Plates 2—7 are unaffected by the conditions that hold at the front of the waves. The figures by which Pearson and Lee's figs. 4—11 of their Plates 1 and 2 should be replaced are the figures numbered 4—11 on Plates 2—5 accompanying this paper.

In these figures the fine continuous circle represents the wave-front at the time $t\,[=t' - (0{\cdot}24)2\tau]$. The discontinuity of the electric field at the wave-front is shown by the change of direction of the lines of force at this circle. Those lines of force which are determined by putting $\dfrac{Q\lambda}{2\pi El}$ equal to $\pm\frac{1}{100}$, $\pm\frac{1}{10}$, $\pm\frac{3}{10}$, and $\pm\frac{1}{2}$, are shown by the heavy dotted, heavy continuous, fine dotted, and fine continuous lines respectively. The dotted circles that lie within the fine continuous circle are curves at which Q vanishes, or the electric force has no radial component. A surface $Q = 0$ travels outwards at a varying rate so as to lie within the wave-front $r = Ct$ and to tend to overtake it as t increases. This is shown by the inner dotted circles in figs. 5—8, and by the outer dotted circles which in figs. 9—11 lie within the fine continuous circle. It appears that no spherical surface of the set given by $Q = 0$ is the front of the advancing wave-train, but that one of these surfaces tends to coincidence with this front as the wave-train advances.

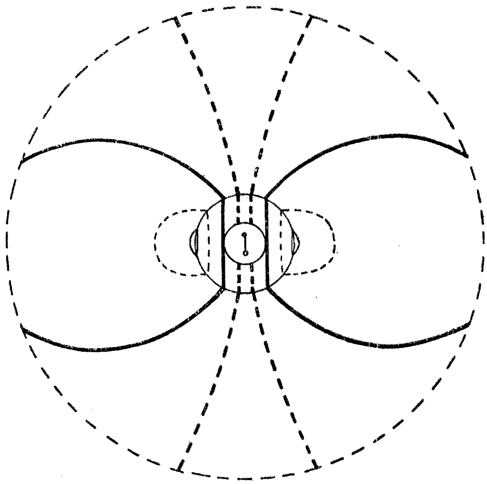
The discontinuity of the electromagnetic field may also be shown in a striking manner by tracing curves to represent at particular instants the values of the transverse component Θ of the electric force, which correspond with all values of r, the distance of a point from the oscillator. Consider points in the equatorial plane of the oscillator, for which $\theta = \frac{1}{2}\pi$. The form of Θ as a function of r is determined by the equations

$$\Theta = \frac{A}{r^3}e^{-\frac{\nu}{\lambda}(Ct-r)}\left[\left(1 - \frac{r\nu}{\lambda} - \frac{r^2(4\pi^2-\nu^2)}{\lambda^2}\right)\sin\frac{2\pi}{\lambda}(Ct - r + \epsilon) + \frac{2\pi r}{\lambda}\left(1 - \frac{2r\nu}{\lambda}\right)\cos\frac{2\pi}{\lambda}(Ct - r + \epsilon)\right],$$

when $Ct > r$, and

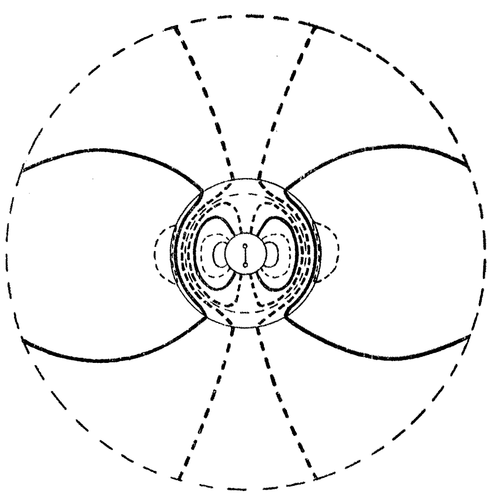
$$\Theta = \frac{A}{r^3}\sin\frac{2\pi\epsilon}{\lambda},$$

Fig. 4.



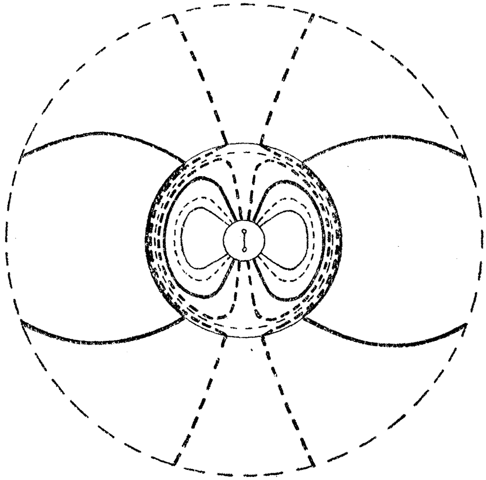
$t = (0{\cdot}26)\, 2\tau.$

Fig. 5.



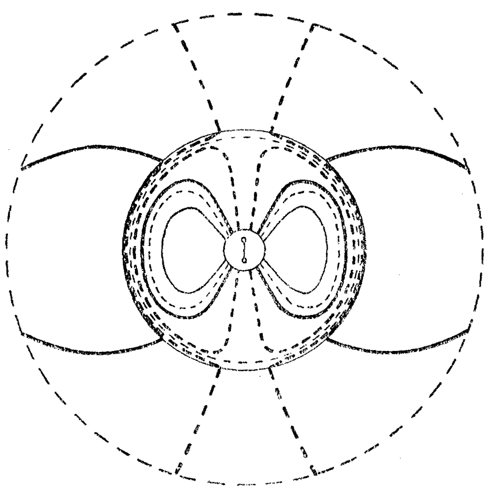
$t = (0{\cdot}385)\, 2\tau.$

FIG. 6.



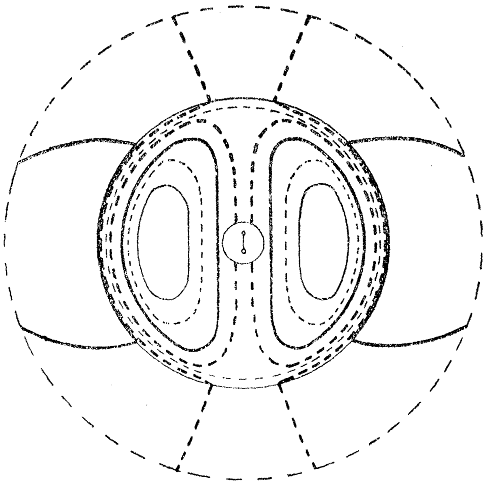
$t = (0{\cdot}51)\,2\tau.$

FIG. 7.



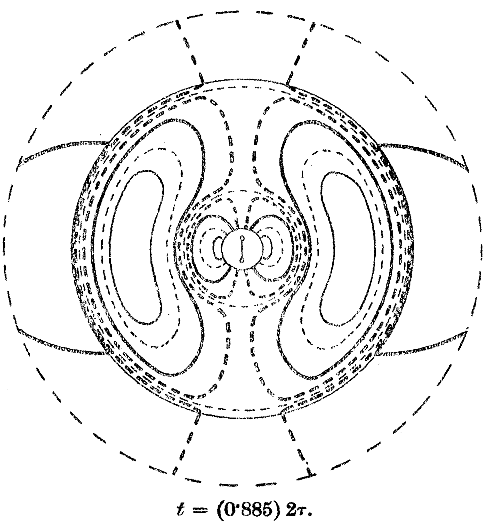
$t = (0{\cdot}635)\,2\tau.$

FIG. 8.



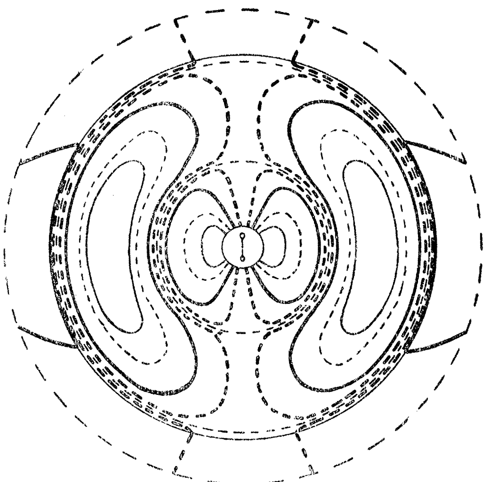
$t = (0{\cdot}76)\, 2\tau.$

FIG. 9.



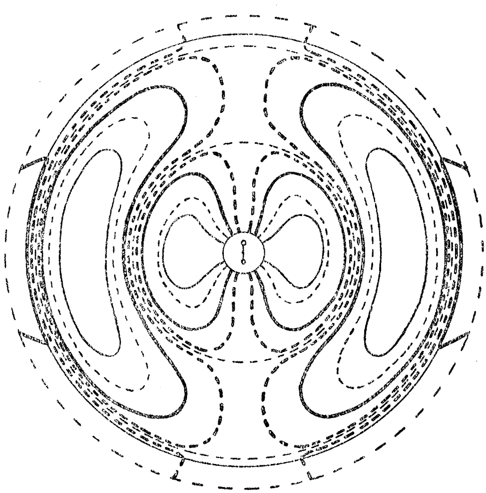
$t = (0{\cdot}885)\, 2\tau.$

FIG. 10.



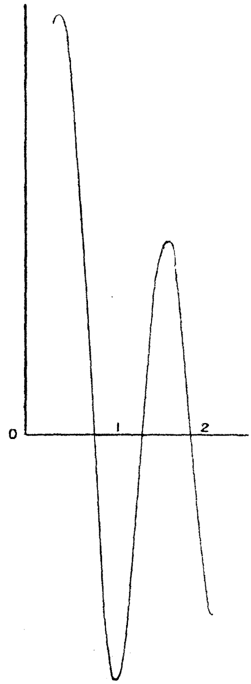
$t = (1{\cdot}01)\,2\tau.$

FIG. 11.



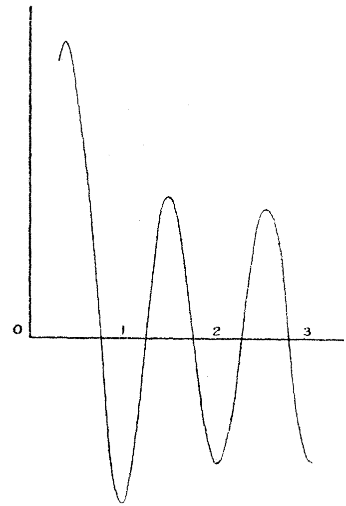
$t = (1{\cdot}135)\,2\tau.$

FIG. 1



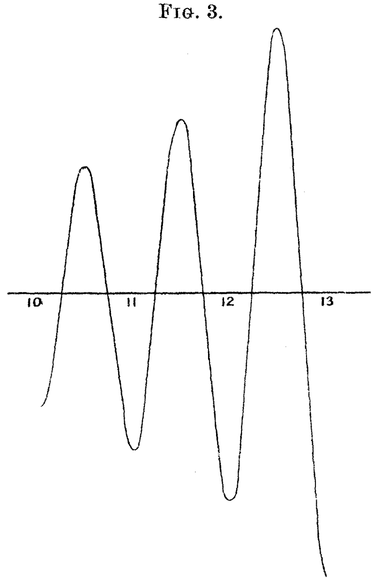
Front of waves after two periods.

FIG. 2.



Front of waves after three periods.

FIG. 3.



Front of waves after thirteen periods.

when $\text{C}t<r$. Figs. 1—3 of Plate 6, represent parts of the curves, of which Θ is ordinate and r is abscissa, at instants which are the ends of the second, third, and thirteenth periods from the beginning of the vibrations.* The parts of the curves in figs. 1 and 2, which are very near the oscillator, are omitted, and, in the three curves, the parts which lie beyond the advancing wave-fronts are indistinguishable from the axis of abscissæ. In figs. 1 and 2, as originally drawn, A was given the value 0·1, and ν and ϵ have the same values as in the previous discussion. In fig. 3 A was given the value 1. The curve in fig. 3 has been drawn for values of r between $r = 10\lambda$ and $r = 13\lambda$. In each case the terminal point of the curve towards the right represents the value of Θ at the advancing wave-front at the instant in question. Near the oscillator, the maxima and minima values of Θ diminish as the distance of them from the oscillator increases, as is shown in figs. 1 and 2. This is due to the preponderance of the factor $1/r^3$ when r is small. When the front of the train of waves has travelled over as few as three wave-lengths, this tendency is already checked by the tendency of the factor $e^{\nu r/\lambda}$ to increase with r, as is seen in fig. 2, where the last minimum is almost exactly equal to the previous maximum. When the front of the train of waves has travelled over a larger number of wave-lengths, the maxima and minima near the front exceed those at a little distance behind the front, as is shown in fig. 3, where there is a regular increase in the maxima and minima values as the front of the train of waves is approached. A comparison of figs. 1 and 2 with each other shows the diminution of the maxima and minima at the same places as time goes on. This is due to the damping of the oscillations by radiation. The same comparison shows also that the maxima and minima near the front of the train of waves do not suffer diminution to the same extent, and the same thing is shown by comparing fig. 3 with these, allowance being made for the difference of scale. In fact, the disturbance at the front of the wave-train suffers diminution through spherical divergence only, for the factor $e^{-\nu(\text{C}t-r)/\lambda}$ has the value unity at the front of the waves, and, when r is at all large, the value of Θ at the front is very nearly equal to

$$-\text{A}\sin(2\pi\epsilon/\lambda)(4\pi^2+\nu^2)/\lambda^2 r,$$

so that it is very nearly proportional to r^{-1}.

* In the arithmetical work which is requisite for tracing these curves and in some of the remaining arithmetical work of the paper, I had the good fortune to secure the collaboration of Mr. J. W. Sharpe, formerly Fellow of Gonville and Caius College, Cambridge, who made the necessary calculations. The paper is much more complete than it would have been without his help.